Universidade Federal do ABC

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Study of tumours via Lie symmetries

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LIGIA CORRÊA DE SOUZA

STUDY OF TUMOURS VIA LIE SYMMETRIES

Doctorate thesis presented to Programa de Pós-Graduação em Matemática in fulfillment of the thesis requirement for the degree of Doctor in Mathematics.

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DEDICATION

To Julio - who cancer took too soon from this world.

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At first it was hard. In the end it felt like I was at the beginning.
— Unknown. And Barbara Müller.

RESUMO

No presente trabalho apresentamos os resultados relacionados ao modelo matemático que descreve a invasão de células tumorais no tecido circundante. O modelo consiste em um sistema de equações diferenciais parciais e focana interação entre as células tumorais e o tecido circundante. Para analisar as soluções do sistema, aplicamos a teoria de Simetrias de Lie. Como resultado, apresentamos todos os geradores associados ao grupo de transformações, algumas soluções invariantes encontradas e a análise biológica de soluções particulares.

Palavras-chave: Câncer, Simetrias de Lie, Modelo matemático.

ABSTRACT

In this work we present results related to a mathematical model describing the invasion of tumor cells in a host tissue. The model consists of a system of partial differential equations and focuses on the interaction among tumor cells and the host tissue. In order to analyze the system solutions, we apply the theory of Lie symmetries. As a result, we present all generators from the associated group of transformation and some invariant solutions that were found and the biological analysis of particular solutions.

Keywords: Cancer, Lie symmetries, Mathematical model.

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1 INTRODUCTION

Noncommunicable diseases (*NCDs*), also known as chronic diseases, are responsible for killing approximately 41 million people each year according to the World Health Organization (*WHO*) [19]. Although those diseases are frequently associated only with ageing, evidence has shown that nearly 17 million of them are considered premature and disproportionately occur in the poorest countries.

NCDs usually result in long-term health consequences and often require a long-term treatment, typically caused by unhealthy behaviours, but can also result from a combination of genetic, physiological and environmental factors. The International Federation of Red Cross and Red Crescent Societies (*IFRC*), the world's largest humanitarian network, states cardiovascular diseases, chronic respiratory diseases, diabetes and cancers as responsible for over two thirds of deaths globally, being the latter one as the second leading cause [17].

WHO also estimates that between 30% to 50% of all cancers can currently be prevented by avoiding risk factors and implementing prevention strategies [19]. Furthermore, early detection of cancer and appropriate treatment and care of patients who develop cancer minimize its burden. Towards that way, knowing and understanding how cancer spreads is crucial for the global fight against it.

Due to the importance of analyzing the spread of cancer, many models have been developed with different approaches focusing on a variety of cancer types and stages ([1, 2, 3, 4, 5, 9, 13, 22]).

As in as [3], this thesis focuses on the avascular stage of a solid tumour modelling the interaction among cancer cells, the extracellular matrix and the matrix-degrading enzyme using a system of partial differential equations. Mainly we use a generalization of the system in [3] and resolve it analytically.

One of the ways to find solutions to differential equation is by Lie symmetries [7]. Lie's theory chiefly treats Lie groups of point symmetries, which are completely characterized by infinitesimal generators and by them we are able to construct solutions of partial differential equations. All the details on how to develop these solutions and what conditions to verify and assume are presented in the following chapters.

As far as we know, only [8] and [9] found analytical solutions for this kind of model, also using Lie symmetries, but for a system with 3 independent variables and constant diffusions.

In this thesis we propose a generalization of the model in [3] and analyze some solutions found by Lie symmetries. This work is divided into six chapters. Chapter 2 presents basic facts about cancer, its growing and spreading dynamic, a few data related to it and the mathematical model to be studied. In chapter 3 we show a summary of the applied theory with examples strictly constructed from the results obtained in the present work. We carry out in chapter 4 a complete group classification of the Lie point symmetries of the system proposed. In chapter 5 we obtain and proceed an in-depth analysis of some invariant solutions of the model. In the last chapter we present concluding remarks about the results and future perspectives related to it.

2 CANCER

Worldwide, chronic diseases are responsible for almost 70% of all deaths, according to the WHO, which includes cancer as the second leading cause, estimated to account for 9.6 million deaths in 2018 and over 10 million in 2020.

Cancer is the name given to a set of more than 200 diseases having in common a disordered growth of cells, invading tissues and organs [11], see Figure 1 [1].

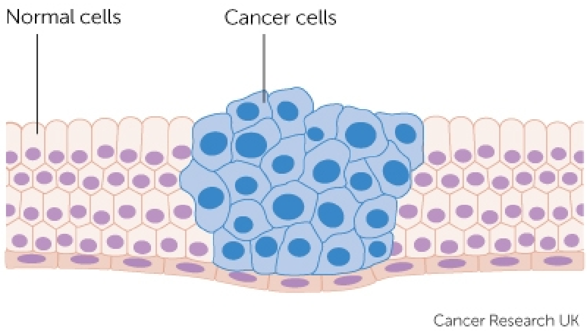

Figure 1: Emergence of tumour cells.
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These diseases can be classified according to the locus they start in the body, such as breast cancer or prostate cancer, known as primary tumour [26]. We can also group cancer according to the type of cell they start in. There are 5 main groups:

- carcinoma: begins in the skin or in tissues that line or cover internal organs. There are different subtypes, including adenocarcinoma, basal cell carcinoma, squamous cell carcinoma and transitional cell carcinoma;
- sarcoma: begins in the connective or supportive tissues such as bone, cartilage, fat, muscle or blood vessels;
- lymphoma and myeloma: begin in the cells of the immune system;

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- leukaemia: this is a cancer of the white blood cells. It starts in the tissues that make blood cells such as the bone marrow;
- brain and spinal cord cancers: known as central nervous system cancers.

Growing abnormally, cells can evolve to a tumour mass which can be classified as benign or malignant, the latter one known as cancer. According to [8], among all cancer types, solid tumours cause 80% of all deaths and their growth occur in two different stages: avascular and vascular. Cancer can sometimes spread to other parts of the body – this is called a secondary tumour or a metastasis, which is an important stage to analyse from a biological point of view although the avascular stage is the focus of our study and it will be explained in detail in the following section.

2.1 CANCER GROWING AND SPREADING

As all our body cells, cancer cells continue to grow encapsulated within a membrane called *basement membrane*. As we can see in Figure 2 [2], as cancer cells grow, the blood vessels get further away, and because of their need for nutrients and oxygen to live, these cells send out signals to trigger the growth of new blood vessels, called *capillaries*, within the tumour, culminating in a process known as *angiogenesis*.

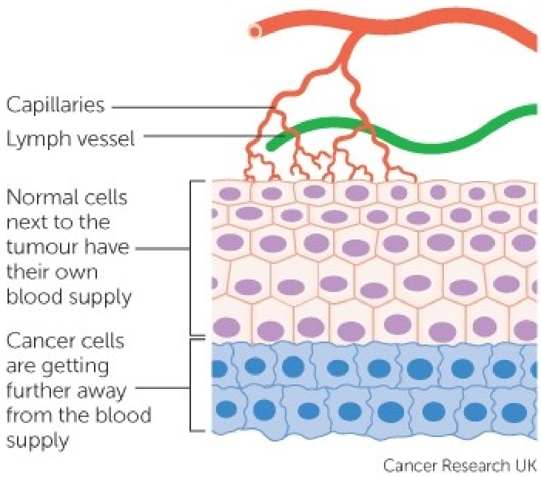

Figure 2: Growing of tumour cells.
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Now with a blood supply of their own, cancer cells can grow even bigger and throughout some blood vessels or the lymphatic system. They can spread and become new tumours themselves. This may occur due to tumour cells behavior that, unlike health body cells, tends to produce substances which stimulate their movement and the breaking of the membrane that contains them, as illustrated in Figure 3 [3], culminating at the vascular stage.

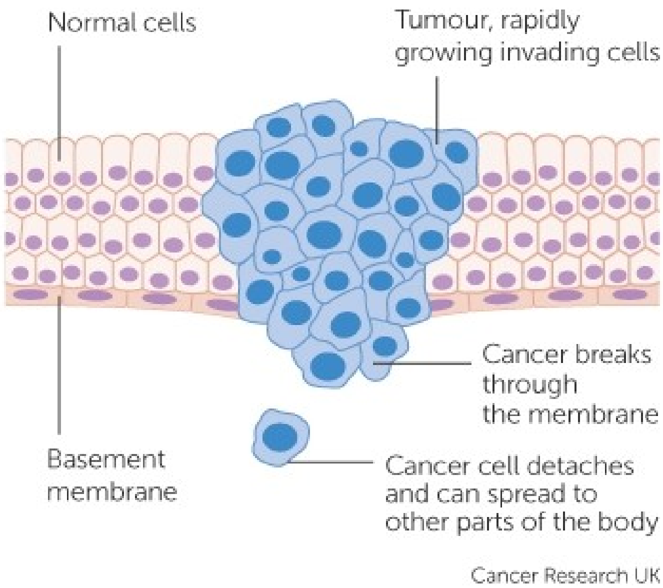

Figure 3: Spreading of tumour cells.
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Although this process may seem fruitful and easily executed, actually it is a complicated roll of steps where many cancer cells die during its evolution.

It is known that tumours can spread into some tissues more easily than others, which may be related to how circulatory system works, as succinctly represented in Figure 4[4].

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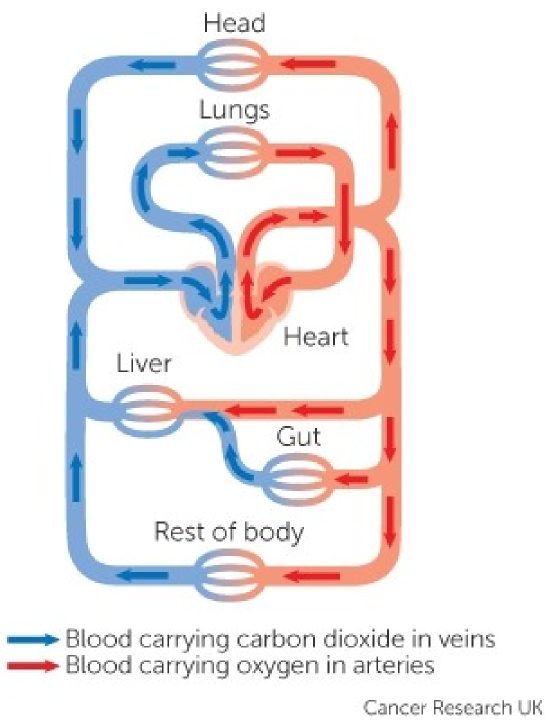

Figure 4: Blood path in circulatory system succinctly represented.
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https://www.cancerresearchuk.org.

For instance, cancers of the large bowel often spread to the liver, which may happen since blood circulates from the bowel through the liver on its way back to the heart.

2.2 DATA

Each year, approximately 400.000 children aged 19 or younger develop cancer worldwide. The most common cancers vary between countries, but leukemia leads the occurrences in Figure 5[5].

5 The designations employed and the presentation of the material in this publication do not imply the expression of any opinion whatsoever on the part of the World Health Organization / International Agency for Research on Cancer concerning the legal status of any country, territory, city or area or of its authorities, or concerning the delimitation of its frontiers or boundaries. Dotted and dashed lines on maps represent approximate borderlines for which there may not yet be full agreement.

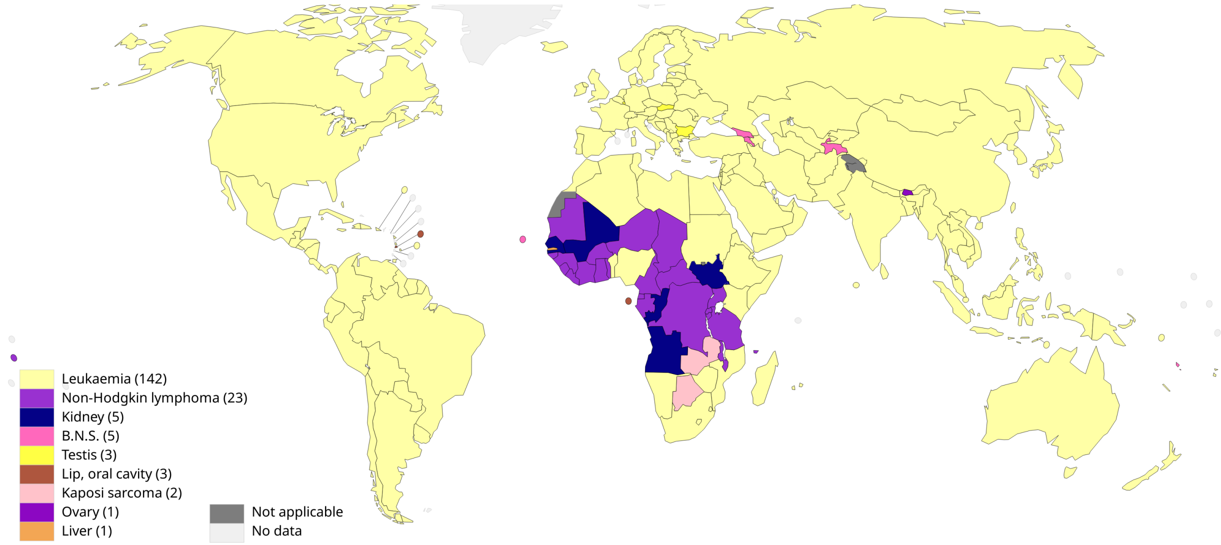

Figure 5: Top cancer per country, estimated age-standardized incidence rates in 2020, both sexes, ages 0-19.
Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/today/home. Information taken 20 apr. 22.

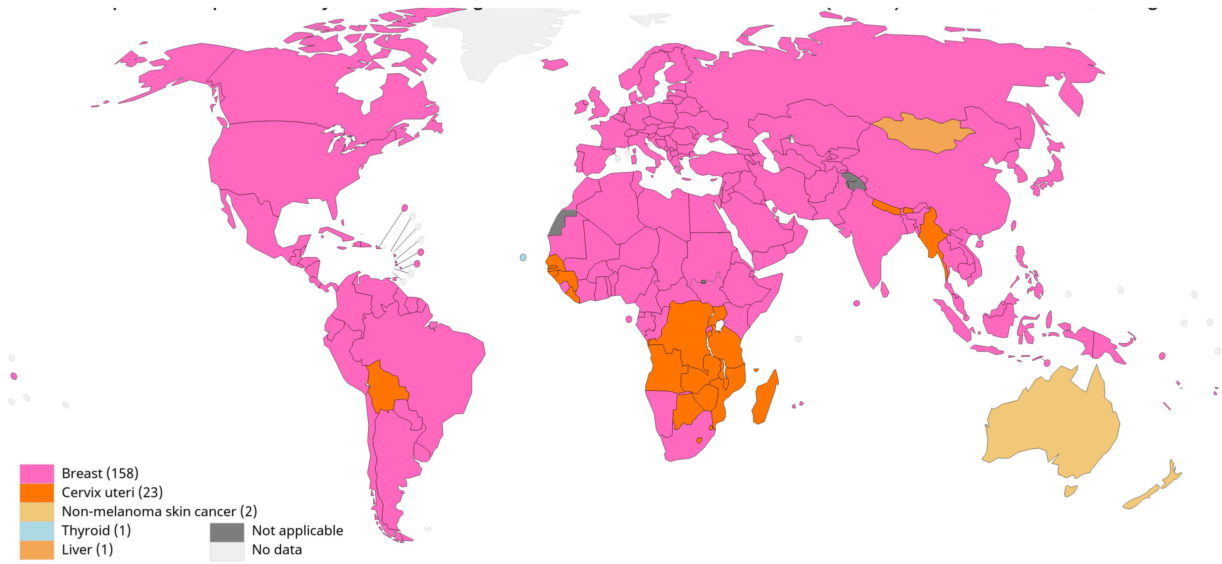

Figure 6: Top cancer per country, estimated age-standardized incidence rates in 2020, females, all ages.
Source: Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/today/home. Information taken 18 nov. 21.

Figures 6 and 7 [6] show the most common cancer in 2020 in each country and the most deadly one, respectively, considering females without age range.

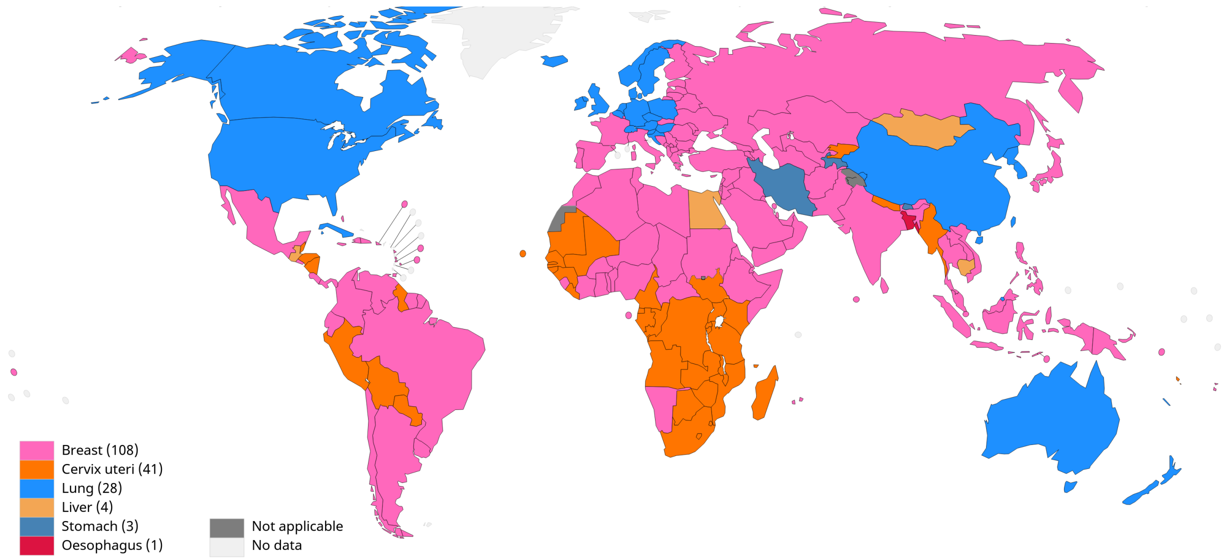

Figure 7: Top cancer per country, estimated age-standardized mortality rates in 2020, females, all ages.

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Figures 8 and 9 [6] present the same date but about males, also in 2020.

Comparing both genres we can observe that breast, prostate, lung and cervix uteri cancers represent the majority rates of incidence and mortality in 2020.

> "The incidence of cancer rises dramatically with age, most likely due to a build-up of risks for specific cancers that increase with age. The overall risk accumulation is combined with the tendency for cellular repair mechanisms to be less effective as a person grows older." [19, inicial page].

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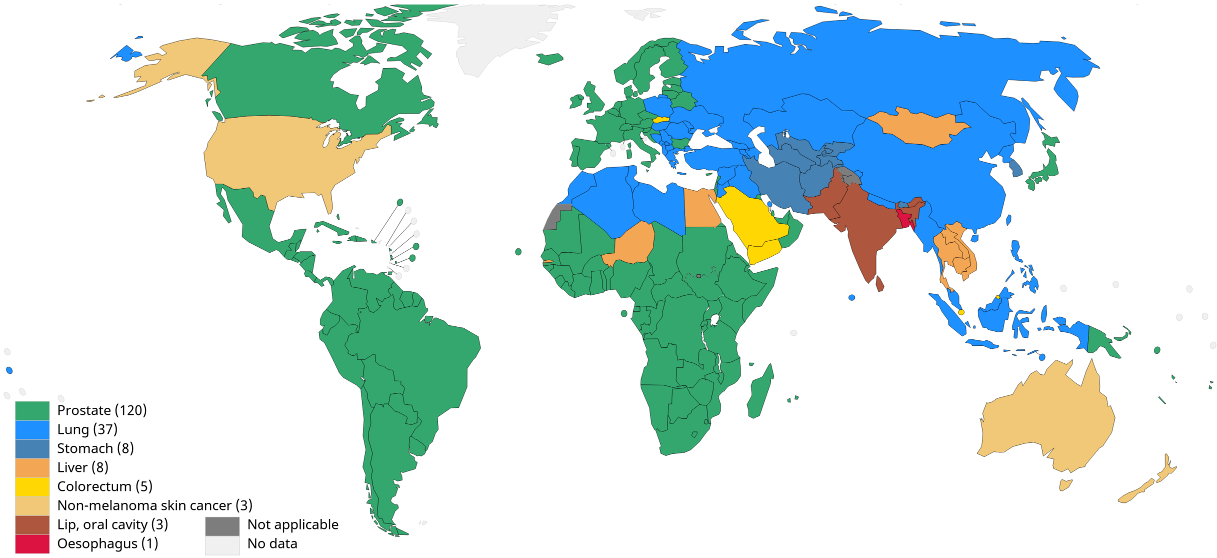

Figure 8: Top cancer per country, estimated age-standardized incidence rates in 2020, males, all ages.

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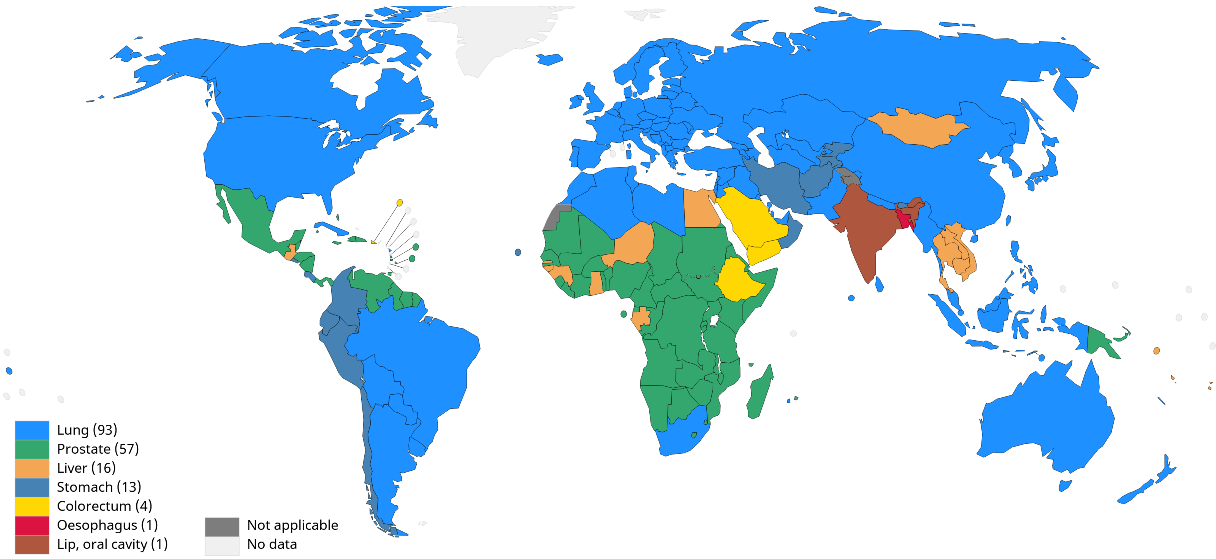

Figure 9: Top cancer per country, estimated age-standardized mortality rates in 2020, males, all ages.

Source: Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/today/home. Information taken 18 nov. 21.

Figures 10 and 11[7] combined with the aforementioned information lead us to conclude that countries with higher life expectancy tend to have a higher rate of elderly people dying from cancer than other countries, simply because they have a greater number of elderly people among their inhabitants.

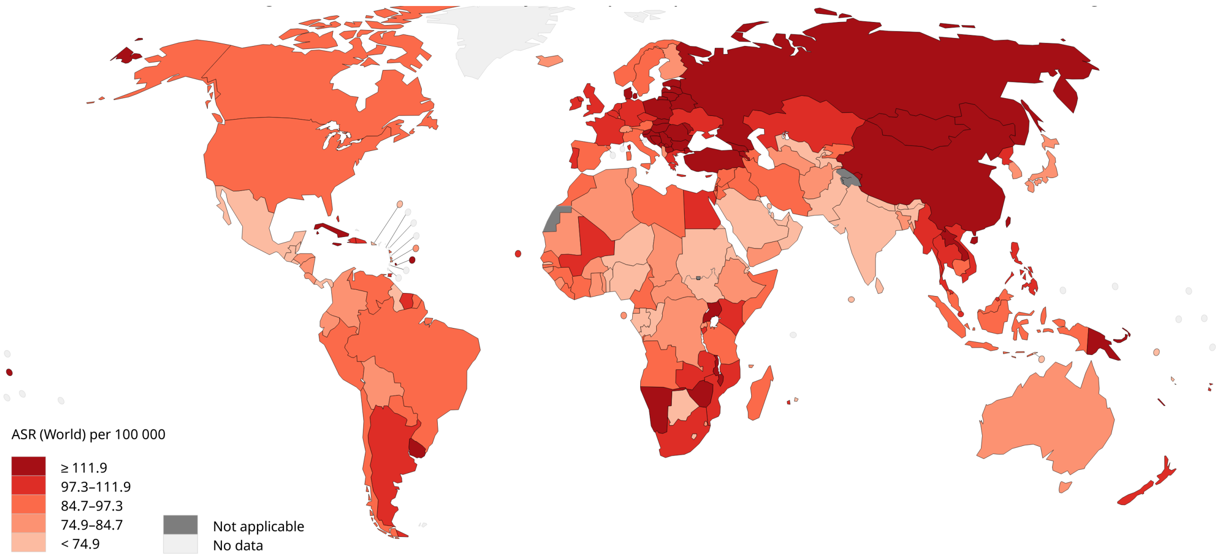

Figure 10: Estimated age-standardized mortality rates in 2020, all cancers, both sexes, all ages.
Source: Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/today/home. Information taken 20 apr. 22.

Considering Figure 11, we can observe that young people, who have fewer risk factors for developing cancer such as smoking for many years or having lower immunity due to age and other diseases, are more likely to die from cancer in the poorest countries. This is probably due to the low investment in health, since there is a high cost of treatment, such as hospital facilities and medicines. Thus, cancer also exposes a worldwide social problem.

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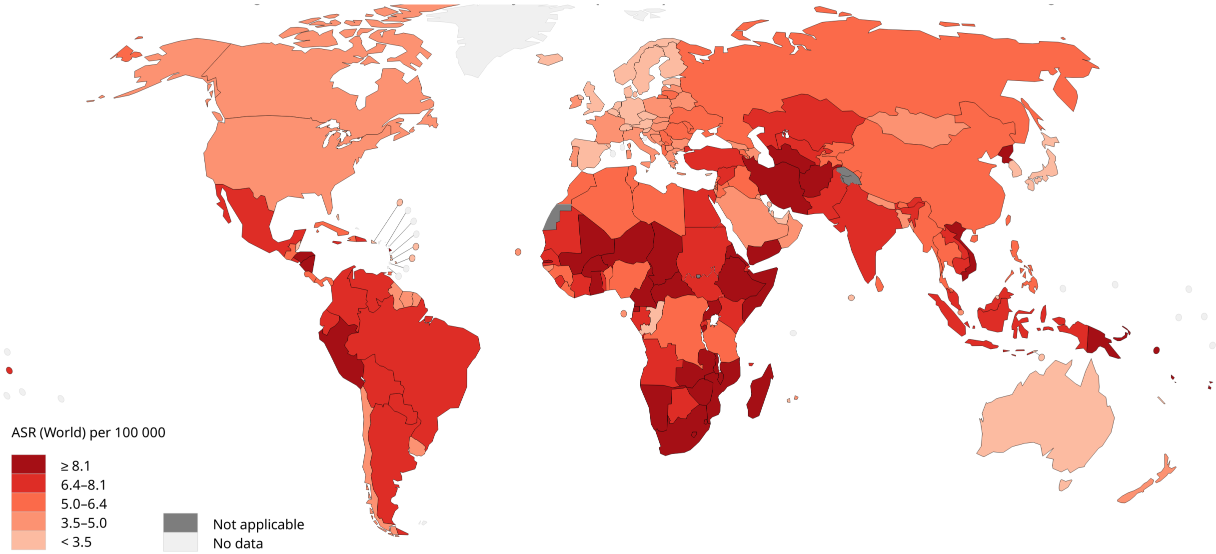

Figure 11: Estimated age-standardized mortality rates in 2020, all cancers, both sexes, ages $0-34$.
Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/today/home. Information taken 20 apr. 22.

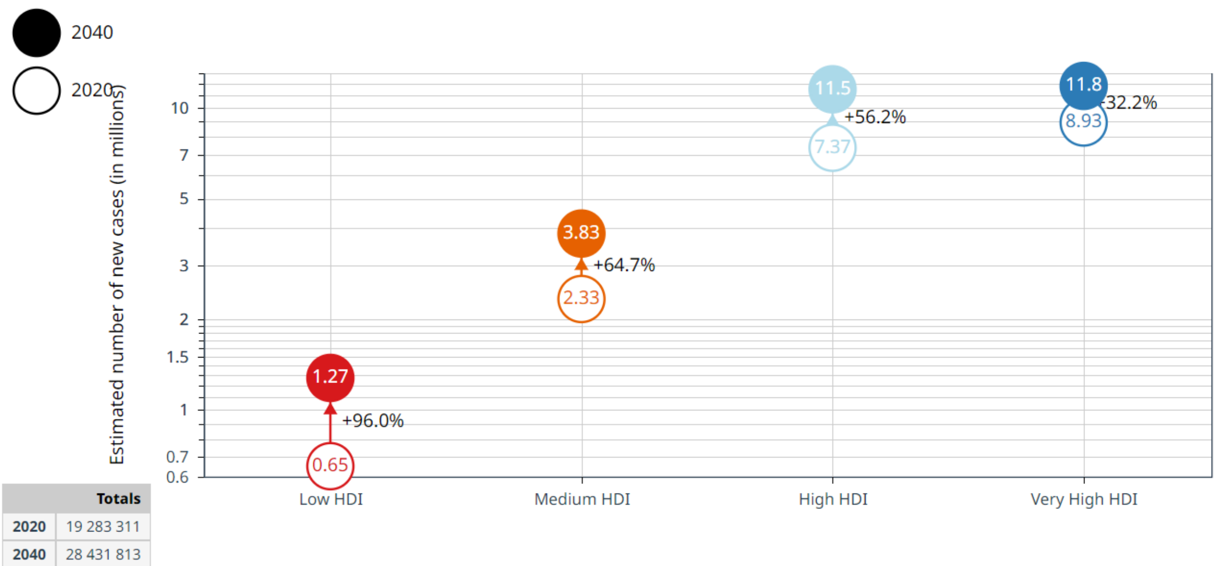

Figure 12: Estimated number of new cases from 2020 to 2040, both sexes, ages $0-85$.
Source: ©International Agency for Research on Cancer. All rights reserved. Map produced by GLOBOCAN 2020. https://gco.iarc.fr/tomorrow. Information taken 23 oct. 22.

In order to support this previous affirmation, Figure 12[8] presents the prediction of rising cancer cases in 2040 comparing the Human Development Index (HDI) worldwide, which is

> "a summary measure of average achievement in key dimensions of human development: a long and healthy life, being knowledgeable and have a decent standard of living." [23, inicial page].

In this work we will focus on modeling the spread of cancer, without considering prevention, treatments and social and economic factors involved, although all of them develop a substantial role in cancer mortality.

2.3 MATHEMATICAL MODEL

Tumour cells produce a number of matrix degradative enzymes (*MDE*) in order to invade the surrounding tissue by diffusion, passing by the degradation of the extracellular matrix (*ECM*), which is a compound of macromolecules including collagens, proteoglycans, and glycoproteins, helping the growth of different tissues to the maintenance of an entire organ. *ECM* can be seen as a set of substances produced and also eliminated by cells. Besides that, the degradation led by cancer forces *ECM* to reorganize itself, leading to haptotaxis – the directed migratory response of tumour cells. This local degradation process of the *ECM* is a critical aspect of the growth and spread of cancer, creating a space where the tumour cells may move by diffusion [9].

In [3] a continuous mathematical model describing the invasion of *ECM* by tumours cells, at the avascular stage and based on solid tumour growth, is presented, considering one and two dimensions of it, and also its discrete version. They are proposed to understand the dynamics of the interactions among the cells in order to predict its behavior, and focus on the macro-scale structure, considering cell population level.

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Over time, different models have been developed by a variety of approaches, and some of these works ([1, 3, 2, 4, 5, 9, 13, 22]) are summarized into the following timeline presented in Figure 13:

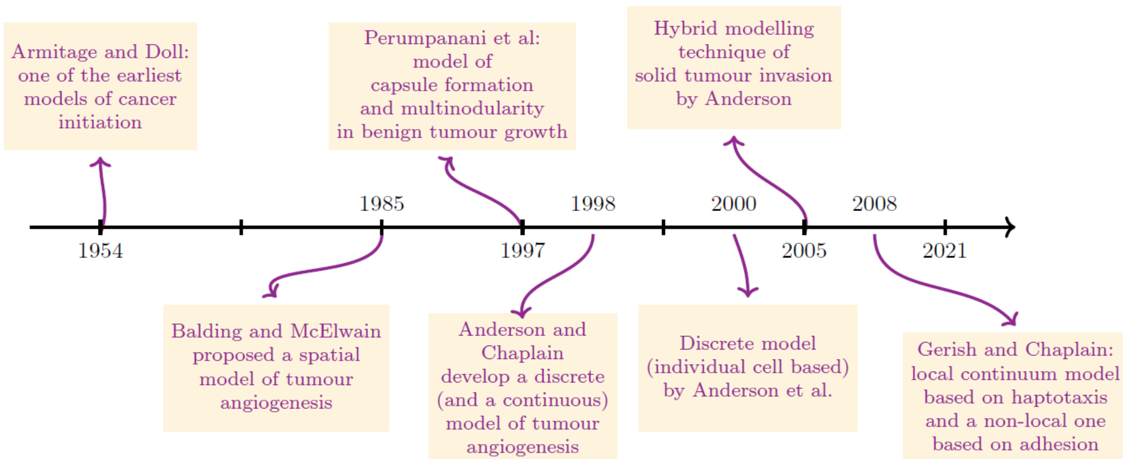


Figure 13: Simplified timeline with some of the key work in the cancer model literature. Source: The author.

Our work considers the 1-dimensional continuous model introduced in [3] and, for that, as in [3], we have three dependent variables of time t and space x: cancer cells density, density of the extracellular matrix (ECM) and concentration of a generic matrix-degrading enzyme (MDE), which ones are represented by $N(x,t)$, $E(x,t)$ and $M(x,t)$, respectively.

As for models for population dispersal, in general, cancer cells movement is driven by random motility with flux $J_{random} = -D\nabla N$, where the cell random motility coefficient $D > 0$ can, generally, be function of time t, space x, and the solution (E, M, N) ([3, 13, 15]).

Although not widely confirmed *in vivo* situation, it is reasonable to assume that cancer cells movement is also driven by a haptotactic response to ECM gradients [3]. Recent works have shown that cancer cells frequently exhibit cell migration behaviour guided by gradients of some surfaces such as the ECM, *in vivo* and *in vitro* situations [10, 14, 20, 21].

In this model, cancer cells proliferation is deliberately left aside so the haptotaxis can be properly investigated. According to [3], the haptotaxis flux is taken by $J_{haptotaxis} =$

$\rho N \nabla E$, where the constant ρ is the haptotatic coefficient and is assumed to be non-negative. Then we arrive at the conservation equation for the tumour cell density:

$$\frac{\partial N}{\partial t} + \nabla(J_{random} + J_{haptotaxis}) = 0.$$

Hence, considering a 1-dimensional model, we arrive at the cell density equation (1):

$$N_t = \underbrace{(DN_x)_x}_{diffusion} - \underbrace{\rho(NE_x)_x}_{haptotaxis}. \tag{1}$$

As a non-motile matter, the ECM changes merely through its local degradation by MDE upon contact at a positive rate δ, assuming there is no matrix remodelling by cells, which is completely possible according to the literature [3]. Altogether, these yield the following evolution equation for the ECM:

$$E_t = \underbrace{-\delta ME}_{degradation}. \tag{2}$$

At last, the MDE is assumed to diffuse (D_2) freely in the spatial domain, where enzymes are released at a constant rate μ by the cells and are removed from the system at a constant rate λ. The latter one happens as a natural decay and also by deactivation of the enzymes and for simplicity we assume that there is a linear relationship between the density of tumour cells and the level of active MDE in the surrounding tissues. Considering MDE diffusion (D_2) as a constant, the evolution equation for the MDE concentration holds

$$M_t = \underbrace{D_2 M_{xx}}_{diffusion} + \underbrace{\mu N}_{enzyme\ production} - \underbrace{\lambda M}_{decay}. \tag{3}$$

Equations (1), (2) and (3) give us the system (4):

$$\begin{cases} N_t = \underbrace{(DN_x)_x}_{diffusion} - \underbrace{\rho(NE_x)_x}_{haptotaxis}, \\ E_t = \underbrace{-\delta ME}_{degradation}, \\ M_t = \underbrace{D_2 M_{xx}}_{diffusion} + \underbrace{\mu N}_{enzyme\ production} - \underbrace{\lambda M}_{decay}. \end{cases} \tag{4}$$

2.3.1 Cancer cells diffusion

In [3] the authors studied the solutions of system (4) numerically considering D constant and also presented simulations assuming that the diffusion D is directly proportional to MDE concentration.

An interesting review of models of brain cancer spreading in [16] indicates that diffusion reasonably models the cell spreading dynamics observed *in vitro* experiences. *In vivo* studies with rats state diffusion of brain cancer cells in white matter different from that in the grey matter cells. So, [16] presents a model of glioma – neoplasm of neural cells capable of division – invasion by taking the diffusion D to be a function of the spatial variable taking into account the spatial heterogeneity of brain cells, i.e., with non-constant diffusion.

The effort to enlarge the dependence of tumour cells with other model variables is valid, considering that cancers spread into nearby tissues also by the cells directly moving. According to [26], about a couple years ago scientists discovered a substance made by cancer cells which stimulates them to move, which might be involved in the local spread of cancers.

In [15, page 402] one extension of the classical diffusion model for insect dispersal is presented: due to population pressure, diffusion increases depending on the population density at a given time. That is, the flux $\mathbf{J}$ is given by

$$\mathbf{J} = -D(N)\nabla N, \quad \frac{dD}{dN} > 0.$$

Also in [15, page 402] is presented a typical form for $D(N)$ as $D_0(N/N_0)^p$, with $p > 0$ and D_0 and N_0 positive constants. Considering one dimension for this case of insect dispersal suffering the population pressure, we have

$$N_t = D_0\left(\left(\frac{N}{N_0}\right)^p N_x\right)_x,$$

which is equivalent to porous media equation [15]. Notice that the solution to this is fundamentally different when diffusion is a constant – where there exists diffusion even though there is no tumour.

Particularly related to tumours, [22] assumed a non-constant diffusion with dependence on the tumour given by $D = D_1 N$ with D_1 constant.

In view of these, and taking into account [15, 22, 26], we consider a general diffusion dependence of the type $D = D_1 N^p$, with D_1 a constant and $p \in \mathbb{R}_+$. Here, pressure-dependent diffusion may vary with tumor type, so the parameter p can represent this. Then (1) becomes

$$N_t = \underbrace{(D_1 N^p N_x)_x}_{diffusion} - \underbrace{\rho(N E_x)_x}_{haptotaxis}.$$

Hence, the system (4) can be rewritten and is equivalent to

$$\begin{cases} N_t &= D_1(N^p N_{xx} + pN^{p-1}N_x^2) - \rho(N_x E_x + N E_{xx}), \\ E_t &= -\delta M E, \\ M_t &= D_2 M_{xx} + \mu N - \lambda M. \end{cases} \tag{5}$$

In the present work we investigate the solutions of (5), a generalization of the model described in [3], from the point of view of Lie symmetries [7], which is a powerful tool to look for analytical solutions of equations.

Before presenting the solutions found and the theory that supports them, we briefly explain in the following section the parameters we used considering biological aspects.

We would like to highlight that the system (5) is already written in a dimensionless form [see 3]. More details about this can also be seen in the following section.

2.4 BIOLOGICAL PARAMETERS

The system given in (5) is considered to hold on some spatial domain Ω (a region of tissue) with appropriate initial conditions for each variable. We assume that tumour cells, *ECM* and *MDE* remain within the domain of tissue under consideration and therefore no-flux boundary conditions are imposed on $\partial\Omega$, the boundary of Ω [1].

Letting

$$\tilde{t} = \frac{t}{\tau}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{N}(\tilde{t}, \tilde{x}) = \frac{N(t,x)}{N_0}, \quad \tilde{E}(\tilde{t}, \tilde{x}) = \frac{E(t,x)}{E_0}, \quad \tilde{M}(\tilde{t}, \tilde{x}) = \frac{M(t,x)}{M_0}, \tag{6}$$

where L corresponds to the maximum invasion distance at the early stage of tumour invasion, we non-dimensionalize the system (5).

Based on [3] the constants in (6) are given by

$$L \in [0.1, 1] \text{ cm}, \quad \tau = \frac{L^2}{D'},$$

where $D' \approx 10^{-6}\text{cm}^2\text{s}^{-1}$ is a chemical diffusion coefficient previous estimated and at this work we chose $L = 1$cm.

The Table (1) shows the parameters u of the dimensional system, the correspondents non-dimensional parameters $\tilde{u}$ and a brief description of each one of them, all constants.

Table 1: Parameters u of dimensional system, the correspondents non-dimensional parameters $\tilde{u}$ and a brief description of each one of them.

u	unit	$\tilde{u}$	**Description**
D_1	cm^2s^{-1}	$\tilde{D_1} = \frac{D_1}{D'}$	cancer cell motility coefficient
ρ	$cm^2s^{-1}nM^{-1}$	$\tilde{\rho} = \frac{E_0\rho}{D'}$	haptotactic coefficient
δ	$nM^{-1}s^{-1}$	$\tilde{\delta} = \tau M_0 \delta$	cancer cell proliferation rate
D_2	cm^2s^{-1}	$\tilde{D_2} = \frac{D_2}{D'}$	*MDE* diffusion coefficient
μ	s^{-1}	$\tilde{\mu} = \frac{\tau\mu N_0}{M_0}$	rate of *MDE* release by cancer cells
λ	s^{-1}	$\tilde{\lambda} = \tau\lambda$	*MDE* degradation rate

The haptotactic parameter $\rho \approx 2600$ cm^2 s^{-1} M^{-1} was estimated in [3] and the parameter $E_0 \in [10^{-11}, 10^{-8}]$M was taken from the experiments in [25]. Assuming that a tumour cell has the volume $1.5 \cdot 10^{-8}$ cm^3 hence $N_0 = 6.7 \cdot 10^7$ cells cm^{-3} [1].

Thus the non-dimensional system is given by

$$\begin{cases} \tilde{N}_t &= \tilde{D_1}(\tilde{N}^p \tilde{N}_{xx} + p\tilde{N}^{p-1}\tilde{N}_x^2) - \tilde{\rho}(\tilde{N}_x \tilde{E}_x + \tilde{N}\tilde{E}_{xx}), \\ \tilde{E}_t &= -\tilde{\delta}\tilde{M}\tilde{E}, \\ \tilde{M}_t &= \tilde{D_2}\tilde{M}_{xx} + \tilde{\mu}\tilde{N} - \tilde{\lambda}\tilde{M}. \end{cases} \tag{7}$$

Dropping the tildes in (7) and using Table (1), the non-dimensionalization leads to the system (5).

Regarding to the biological meaning, it is worth mentioning that the parameters μ, λ and δ in system (5) until remain unknown for the *in vivo* situation as same as M_0 [1]. Therefore, those parameters are not obtained from experimental data but estimations supported by the literature, such as in [1], [3] and [9]. A complete list of the values used to analyze the solutions of system (5) is in Table 4, Chapter 5.

Also maintaining the purpose of analyzing biological situations of the system (5) in which there is diffusion of both tumor cells and the matrix of degrading enzymes, we consider $D_1 D_2 \neq 0$.

Although the parameters p, ρ, δ, μ and λ in (5) can be zero, each one of them plays an important biological role in the process of cancer cells invasion. As we stated earlier, while $\rho \neq 0$ indicates the existence of haptotatic movement of cancer cells, $p \neq 0$ is related to tissue heterogeneity in the process of diffusion of cancer cells. On the other hand, if $\delta = 0$, the haptotaxis response has no influence on tumour cells density and ECM remains the same over time. The latter has no biological significance when we are interested in analyzing the avascular stage of cancer.

The last two parameters are more flexible regarding their nullity. $\mu = 0$ eliminates the influence of cancer cells into MDE and so into ECM as well, which means that there is no enzyme production based on cancer cells density and only its natural decay if $\lambda \neq 0$.

3 LIE SYMMETRIES

In order to organize the jumble of solving techniques of ordinary differential equations (*ODE*), until his times, Sophus Lie introduced the idea of a continuous group of transformations, which generated the area known as Lie Theory [6].

Throughout the coming sections, we present part of this theory so as to enable the reader to keep up with the results found. For further details, see [6], [7] and [18]. Furthermore, it is worth saying that, in this chapter, the theorems and propositions are not original.

3.1 LIE GROUPS OF TRANSFORMATIONS

A symmetry group of a system of differential equations can be defined as a group of transformations that apply any solution of the equation to another solution. If a differential equation or system is invariant under the action of a group of Lie point transformations, we can find special solutions constructively called invariant solutions, which are invariant under the action of some subgroup of the admitted total symmetry group by the equation.

Definition 3.1. *A group G is a set of elements with a law of composition ϕ satisfying the following axioms:*

(i) Closure property. For any elements a and b of G, $\phi(a,b)$ is an element of G.

(ii) Associative property. For any elements a, b, c of G we have $\phi(a,\phi(b,c)) = \phi(\phi(a,b),c)$.

(iii) Identity element. There exists a unique identity element e of G such that for any element a of G we have $\phi(a,e) = \phi(e,a) = a$.

(iv) Inverse element. For any element a of G there exists a unique inverse element a^{-1} in G such that $\phi(a,a^{-1}) = \phi(a^{-1},a) = e$.

Definition 3.2. *Let $x = (x_1, x_2, \ldots, x_n)$ lie in a region $D \subset \mathbb{R}^n$. The set of transformations $\overline{x} = X(x; \epsilon)$ defined for each x in D and parameter ϵ in set $S \subset \mathbb{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in S, form a one-parameter group of transformations on D if the following hold:*

(i) *For each ϵ in S the transformations are one-to-one onto D; hence $\overline{x}$ lies in D.*

(ii) *S with the law of composition ϕ forms a group G.*

(iii) *For each x in D, $\overline{x} = x$ when $\epsilon = \epsilon_0$ corresponds to the identity e, i.e., $X(x; \epsilon_0) = x$.*

(iv) *If $\overline{x} = X(x; \epsilon)$, $\overline{\overline{x}} = X(\overline{x}; \delta)$, then $\overline{\overline{x}} = X(x; \phi(\epsilon, \delta))$.*

Definition 3.3. *A one-parameter group of transformations defines a one-parameter Lie group of transformations if, in addition to satisfying axioms $(i) - (iv)$ of Definition 3.2, the following hold:*

(v) *ϵ is a continuous parameter, i.e., S is an interval in $\mathbb{R}$. Without loss of generality, $\epsilon = 0$ corresponds to the identity element e.*

(vi) *X is infinitely differentiable with respect to x in D and an analytic function of ϵ in S.*

(vii) *$\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ, $\epsilon \in S$, $\delta \in S$.*

Example 3.1. *Let $(x, t, E, M, N) \in \Omega \subseteq \mathbb{R}^5$ and $\epsilon \in \mathbb{R}$. So, $(\overline{x}, \overline{t}, \overline{E}, \overline{M}, \overline{N}) = X((x, t, E, M, N); \epsilon) = (x + c\epsilon, t + \epsilon, E, M, N)$ is a one-parameter Lie group of transformations.*

3.2 INFINITESIMAL TRANSFORMATIONS

Consider a one-parameter ϵ Lie group of transformations

$$\overline{x} = X(x, \epsilon) \tag{8}$$

with the identity $\epsilon = 0$ and law of composition ϕ. Expanding (8) about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, we get

$$\overline{x} = x + \epsilon \left(\frac{\partial X(x; \epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} \right) + \frac{1}{2}\epsilon^2 \left(\frac{\partial^2 X(x; \epsilon)}{\partial \epsilon^2} \Big|_{\epsilon=0} \right) + \cdots = x + \epsilon \left(\frac{\partial X(x; \epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} \right) + \mathcal{O}(\epsilon^2).$$

Let

$$\xi(x) = \frac{\partial X(x;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{9}$$

The transformation $x + \epsilon\xi(x)$ is called the *infinitesimal transformation* of the Lie group of transformations (8). The components of $\xi(x)$ are called the *infinitesimals* of (8).

Example 3.2. *In this work we consider the Lie symmetries of (5), with two independent variables (x, t) and three dependent ones (E, M, N).*

A Lie point symmetry of system (5) is a set of transformations with parameter ϵ

$$\begin{aligned}
\overline{x} &= x + \epsilon\xi^1(x,t,E,M,N) + \mathcal{O}(\epsilon^2),\\
\overline{t} &= t + \epsilon\xi^2(x,t,E,M,N) + \mathcal{O}(\epsilon^2),\\
\overline{E} &= E + \epsilon\eta^1(x,t,E,M,N) + \mathcal{O}(\epsilon^2),\\
\overline{M} &= M + \epsilon\eta^2(x,t,E,M,N) + \mathcal{O}(\epsilon^2),\\
\overline{N} &= N + \epsilon\eta^3(x,t,E,M,N) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{10}$$

Example 3.3. *Let $(x,t,E,M,N) \in \Omega \subseteq \mathbb{R}^5$ and $\epsilon \in \mathbb{R}$. Consider the one-parameter Lie group of transformations*

$$\begin{aligned}
(\overline{x},\overline{t},\overline{E},\overline{M},\overline{N}) &= X((x,t,E,M,N);\epsilon)\\
&= \left(\left(\tfrac{\epsilon}{2}+1\right)x, (\epsilon+1)t, E, (1-\epsilon)M, (1-2\epsilon)N\right).
\end{aligned} \tag{11}$$

Notice that the identity of (11) is $\mathbf{0}$ and its law of composition is $\phi(a,b) = a + b$. Using (9), the infinitesimals related to this group are $\xi^1 = \frac{x}{2}, \xi^2 = t, \xi^3 = 0, \xi^4 = -M$ and $\xi^5 = -2N$.

Example 3.4. *Let $(x,t,E,M,N) \in \Omega \subseteq \mathbb{R}^5$ and $\tau \in \mathbb{R}^*_+$. Consider the one-parameter Lie group of transformations*

$$\begin{aligned}
(\overline{x},\overline{t},\overline{E},\overline{M},\overline{N}) &= X((x,t,E,M,N);\tau)\\
&= \left(\left(\frac{e^\tau}{2}+1\right)x, (e^\tau+1)t, E, (1-e^\tau)M, (1-2e^\tau)N\right).
\end{aligned} \tag{12}$$

Notice that the identity of (12) is $\mathbf{1}$ and its law of composition is $\phi(a,b) = ab$. Using (9), considering $\tau = 1$, we have $\xi(x) = \frac{\partial X(x;\tau)}{\partial \tau}\Big|_{\tau=1}$. So, the infinitesimals related to this group are $\xi^1 = \frac{x}{2}, \xi^2 = t, \xi^3 = 0, \xi^4 = -M$ and $\xi^5 = -2N$.

Observe that the infinitesimals of one-parameter Lie groups of transformations shown in equations (11) and (12) are the same. This is no accident. The following theorem can enlighten why.

Theorem 3.1. *(First Fundamental Theorem of Lie) There exists a parameterization $\tau(\epsilon)$ such that the Lie group of transformations (8) is equivalent to the solution of an initial value problem for a system of first-order ODE given by*

$$\frac{d\overline{x}}{d\tau} = \xi(\overline{x}),$$

with $\overline{x} = x$ when $\tau = 0$.

In particular,

$$\tau(\epsilon) = \int_0^\epsilon \Gamma(\epsilon')d\epsilon',$$

where

$$\Gamma(\epsilon) = \frac{\partial\phi(a,b)}{\partial b}\Big|_{(a,b)=(\epsilon^{-1},\epsilon)},$$

$\Gamma(0) = 1$ and 0 means the identity of the group.

Proof: See [6], pages 39 and 40. ∎

Example 3.5. *Consider the one-parameter Lie groups of transformations in examples 3.3 and 3.4. Applying the Theorem 3.1 into the latter one, we obtain:*

$$\Gamma(\epsilon) = \frac{\partial\phi(a,b)}{\partial b}\Big|_{(a,b)=(\epsilon^{-1},\epsilon)} = a\Big|_{a=\epsilon^{-1}} = \frac{1}{\epsilon}$$

and

$$\Gamma(\mathbf{0}) = 1.$$

Thus,

$$\tau(\epsilon) = \int_{\mathbf{0}}^\epsilon \Gamma(\epsilon')d\epsilon' = \ln\epsilon,$$

which implies $\epsilon = e^\tau$. This gives us what we need to parameterize a given group into one in terms of τ with $\phi(a,b) = \tau_1 + \tau_2$.

Without loss of generality, we assume that a one-parameter ϵ Lie group of transformations is parameterized such that its law of composition is given by $\phi(a,b) = a + b$, so that $\epsilon^{-1} = -\epsilon$ and $\Gamma(\epsilon) \equiv 1$. Therefore, we can rewrite (8) as

$$\frac{d\overline{x}}{d\epsilon} = \xi(\overline{x}), \tag{13}$$

with $\overline{x} = x$ when $\epsilon = 0$.

3.2.1 Generators

Definition 3.4. *The infinitesimal generator of the one-parameter Lie group of transformations* (8) *is the operator*

$$X = X(x) := \sum_{i=1}^{n} \xi^i(x)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \xi^i(x)\partial_{x_i}. \tag{14}$$

So, for any differentiable function $F(x) = F(x_1, x_2, \ldots, x_n)$ *one has*

$$XF(x) = \sum_{i=1}^{n} \xi^i(x)\frac{\partial F(x)}{\partial x_i}.$$

It is worth highlighting that we are going to use the notation ∂_{x_i} instead of $\dfrac{\partial}{\partial x_i}$. Moreover, sometimes we prefer to use ξ to represent the infinitesimals related to the independent variables and η to represent the infinitesimals related to the dependent ones.

Thus, the generator related to (5) is given by

$$X = \xi^1\partial_x + \xi^2\partial_t + \eta^1\partial_E + \eta^2\partial_M + \eta^3\partial_N, \tag{15}$$

as we can see in (10).

Theorem 3.2. *The one-parameter* (ϵ) *Lie group of transformations* (8) *is equivalent to*

$$\overline{x} = e^{\epsilon X}x = x + \epsilon Xx + \frac{1}{2}\epsilon^2X^2x + \cdots = \left[1 + \epsilon X + \frac{1}{2}\epsilon^2X^2 + \ldots\right] x = \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}X^k x,$$

where X=X(x) is the operator defined in (14) *and* $X^k = X^k(x)$ *is given by* $X^k = XX^{k-1}$ *for* $k = 1, 2, \ldots$ *and* $X^0F(x) \equiv F(x)$.

Proof: See [6], pages 43 and 44. ∎

Example 3.6. *Consider the infinitesimal generator described by*

$$X = \frac{x}{2}\partial_x + t\partial_t - M\partial_M - 2N\partial_N, \tag{16}$$

which compared to (15) *we have*

$$\xi^1 = \frac{x}{2}, \xi^2 = t, \eta^1 = 0, \eta^2 = -M, \eta^3 = -2N.$$

Using (13), we obtain the following results

$$\frac{d\overline{x}}{d\epsilon} = \frac{\overline{x}}{2} \Rightarrow \frac{2d\overline{x}}{\overline{x}} = d\epsilon \Rightarrow 2\ln\overline{x} = \epsilon + c_0,$$

$$\frac{d\overline{t}}{d\epsilon} = \overline{t} \Rightarrow \frac{d\overline{t}}{\overline{t}} = d\epsilon \Rightarrow \ln\overline{t} = \epsilon + c_1,$$

$$\frac{d\overline{E}}{d\epsilon} = 0 \Rightarrow \overline{E} = c_2,$$

$$\frac{d\overline{M}}{d\epsilon} = -\overline{M} \Rightarrow \frac{d\overline{M}}{\overline{M}} = -d\epsilon \Rightarrow \ln\overline{M} = c_3 - \epsilon,$$

$$\frac{d\overline{N}}{d\epsilon} = -2\overline{N} \Rightarrow \frac{d\overline{N}}{\overline{N}} = -2d\epsilon \Rightarrow \ln\overline{N} = c_4 - 2\epsilon,$$

where c_0, c_1, c_2, c_3 *and* c_4 *are constants with respect to* ϵ*. With* $(\overline{x}, \overline{t}, \overline{E}, \overline{M}, \overline{N}) = (x, t, E, M, N)$ *when* $\epsilon = 0$*, then*

$$(\overline{x}, \overline{t}, \overline{E}, \overline{M}, \overline{N}) = (e^{\frac{\epsilon}{2}}x, e^{\epsilon}t, E, e^{-\epsilon}M, e^{-2\epsilon}N). \tag{17}$$

Example 3.7. *Consider the infinitesimal generator (16). By Theorem 3.2, we can also obtain the one-parameter (*ϵ*) Lie group of transformations related to this generator, as following*

$$X^0x = x, \quad Xx = \frac{x}{2}, \quad X^2x = \frac{x}{4}, \quad X^3x = \frac{x}{8}, \quad \ldots$$

$$\overline{x} = e^{\epsilon X}x = x + \frac{x}{2} + \frac{\epsilon^2}{2!}\frac{x}{4} + \frac{\epsilon^3}{3!}\frac{x}{8} + \cdots = x\left(\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\frac{1}{2^k}\right) = e^{\frac{\epsilon}{2}}x,$$

$$X^0t = t, \quad Xt = t, \quad X^2t = t, \quad X^3x = t, \quad \ldots$$

$$\overline{t} = e^{\epsilon X}t = t + \epsilon t + \frac{1}{2!}\epsilon^2 t + \frac{1}{3!}\epsilon^3 t + \cdots = t\left(\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\right) = e^{\epsilon}t,$$

$$X^0E = E, \quad XE = 0, \quad X^2E = 0, \quad X^3E = 0, \quad \ldots$$

$$\overline{E} = e^{\epsilon X}E = E,$$

$$X^0M = M, \quad XM = -M, \quad X^2M = M, \quad X^3M = -M, \quad \dots$$

$$\overline{M} = e^{\epsilon X}M = M - \epsilon M + \frac{1}{2!}\epsilon^2 M - \frac{1}{3!}\epsilon^3 M + \dots = M\left(\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}(-1)^k\right) = e^{-\epsilon}M,$$

$$X^0N = N, \quad XN = -2N, \quad X^2N = 4N, \quad X^3N = -8N, \quad \dots$$

$$\overline{N} = e^{\epsilon X}N = N - 2\epsilon N + 4\frac{1}{2!}\epsilon^2 N - 8\frac{1}{3!}\epsilon^3 N + \dots = N\left(\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}(-2)^k\right) = e^{-2\epsilon}N.$$

Thus $(\overline{x}, \overline{t}, \overline{E}, \overline{M}, \overline{N}) = (e^{\frac{\epsilon}{2}}x, e^{\epsilon}t, E, e^{-\epsilon}M, e^{-2\epsilon}N)$.

Examples 3.6 and 3.7 have shown two different ways to find a one-parameter (ϵ) Lie group of transformations related to a given generator: the first one use basically the infinitesimals and the latter, only the generator. In both cases we have the essential information to determine the one-parameter (ϵ) Lie group of transformations related to each one. The importance of this fact impacts the path we trace to find the solutions of the system modeling the cancer problem that we chose: in order to find it, at first, we can only determine the generators. The following theory of this chapter completes the further steps.

3.2.2 Invariant functions

Definition 3.5. *An infinitely differentiable function $F(x)$ is an invariant function of the Lie group of transformations (8) if and only if, for any group transformation (8), $F(\overline{x}) = F(x)$. If $F(x)$ is an invariant function of (8), then $F(x)$ is called an invariant of (8) and $F(x)$ is said to be invariant under (8).*

Theorem 3.3. *$F(x)$ is invariant under a Lie group of transformations (8) if and only if $XF(x) \equiv 0$.*

Proof: See [6], page 46. ■

Example 3.8. *Let* $F : \mathbb{R}^5 \to \mathbb{R}$ *described by* $F(x,t,E,M,N) = 2E + tM$. *Consider the infinitesimal generator* (16) *and the Lie group of transformations* (17)*. Then,*

$$F(\overline{x},\overline{t},\overline{E},\overline{M},\overline{N}) = 2\overline{E} + \overline{t}\overline{M} = 2E + e^{\epsilon} te^{-\epsilon} M = 2E + tM = F(x,t,E,M,N).$$

Hence, the function $F(x,t,E,M,N) = 2E + tM$ *is invariant under a Lie group of transformations* (17).

On the other hand,

$$\begin{aligned} XF(x,t,E,M,N) &= \tfrac{x}{2}\tfrac{\partial}{\partial_x}F(x,t,E,M,N) + t\tfrac{\partial}{\partial_t}F(x,t,E,M,N) - M\tfrac{\partial}{\partial_M}F(x,t,E,M,N) \\ &\quad -2N\tfrac{\partial}{\partial_N}F(x,t,E,M,N) \\ &= tM - Mt \\ &= 0. \end{aligned}$$

3.3 EXTENDED TRANSFORMATIONS

Definition 3.6. *A one-parameter* (ϵ) *Lie group of point transformations related to a system* S *is a group of transformations of the form*

$$\begin{aligned} x^* &= X(x,u;\epsilon), \\ u^* &= U(x,u;\epsilon), \end{aligned} \tag{18}$$

acting on the space of $m + n$ *variables*

$$\begin{aligned} x &= (x_1, x_2, x_3, \ldots, x_n), \\ u &= (u^1, u^2, u^3, \ldots, u^m), \end{aligned} \tag{19}$$

where x *represents* n *independent variables and* u *represents* m *dependent ones.*

A Lie group of point transformations (18) admitted by system S leaves S invariant, i. e., the form of S is unchanged in terms of transformed variables for any solution $u = \theta(x)$ of S.

Definition 3.7. *Consider a Lie group of point transformations related to a system* S

$$\begin{aligned} x^{\dagger} &= X(x,u), \\ u^{\dagger} &= U(x,u), \end{aligned} \tag{20}$$

where $x = (x_1, x_2, x_3, \ldots, x_n)$ *represents* n *independent variables and* $u = (u^1, u^2, u^3, \ldots, u^m)$ *represents* m *dependent ones.*

Let

$$u_i^\mu = \frac{\partial u^\mu}{\partial x_i}, \quad (u_i^\mu)^\dagger = \frac{\partial (u^\mu)^\dagger}{\partial x_i^\dagger} = \frac{\partial U^\mu}{\partial X_i}$$

and

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + \cdots + u_{ii_1 i_2 \cdots i_n}^\mu \frac{\partial}{\partial u_{i_1 i_2 \cdots i_n}^\mu} + \cdots, \tag{21}$$

with summation over a repeated index.

D_i *is called total derivative operators.*

Also, let $\partial^k u$ *denotes the set of coordinates*

$$u_{i_1 i_2 \cdots i_k}^\mu = \frac{\partial^k u^\mu}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}},$$

where $\mu = 1, 2, \ldots, m$ *and* $i_j = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$ *corresponding to all kth-order partial derivatives of* u *with respect to* x.

The kth-extended transformation of (20) *is given by*

$$\begin{aligned} x^\dagger &= X(x, u), \\ u^\dagger &= U(x, u), \\ \partial u^\dagger &= \partial U(x, u), \\ &\vdots \\ \partial^k u^\dagger &= \partial^k U(x, u, \partial u, \ldots, \partial^k u), \end{aligned}$$

where $(u_i^\mu)^\dagger$ *of* $\partial u^\dagger$ *are determined by*

$$\begin{pmatrix} (u_1^\mu)^\dagger \\ (u_2^\mu)^\dagger \\ \vdots \\ (u_n^\mu)^\dagger \end{pmatrix} = \begin{pmatrix} U_1^\mu \\ U_2^\mu \\ \vdots \\ U_n^\mu \end{pmatrix} = A^{-1} \begin{pmatrix} D_1 U^\mu \\ D_2 U^\mu \\ \vdots \\ D_n U^\mu \end{pmatrix},$$

A^{-1} *is the assumed existent inverse of the matrix*

$$\begin{pmatrix} D_1 X_1 & D_1 X_2 & \cdots & D_1 X_n \\ D_2 X_1 & D_2 X_2 & \cdots & D_2 X_n \\ \vdots & \vdots & & \vdots \\ D_n X_1 & D_n X_2 & \cdots & D_n X_n \end{pmatrix}$$

and $(u^{\mu}_{i_1 i_2 \dots i_k})^{\dagger}$ *of* $\partial^k u^{\dagger}$ *are determined by*

$$\begin{pmatrix} (u^{\mu}_{i_1 i_2 \cdots i_{k-1} 1})^{\dagger} \\ (u^{\mu}_{i_1 i_2 \cdots i_{k-1} 2})^{\dagger} \\ \vdots \\ (u^{\mu}_{i_1 i_2 \cdots i_{k-1} n})^{\dagger} \end{pmatrix} = \begin{pmatrix} U^{\mu}_{i_1 i_2 \cdots i_{k-1} 1} \\ U^{\mu}_{i_1 i_2 \cdots i_{k-1} 2} \\ \vdots \\ U^{\mu}_{i_1 i_2 \cdots i_{k-1} n} \end{pmatrix} = A^{-1} \begin{pmatrix} D_1 U^{\mu}_{i_1 i_2 \cdots i_{k-1}} \\ D_2 U^{\mu}_{i_1 i_2 \cdots i_{k-1}} \\ \vdots \\ D_n U^{\mu}_{i_1 i_2 \cdots i_{k-1}} \end{pmatrix},$$

$\mu = 1, 2, \dots, m$, $k = 2, 3, \dots, n$ *and* $i_j = 1, 2, \dots, n$, $j = 1, 2, \dots, k$.

Definition 3.8. *Consider the one-parameter* (ϵ) *Lie group of point transformations related to a system* S *in* (18) *and* (19).

The kth-extended transformation of (18) *is given by*

$$\begin{aligned} \bar{x}_i = X_i(x, u; \epsilon) &= x_i + \epsilon \xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u^{\mu}} = U^{\mu}(x, u; \epsilon) &= u^{\mu} + \epsilon \eta^{\mu}(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u^{\mu}_i} = U^{\mu}_i(x, u, \partial u; \epsilon) &= u^{\mu}_i + \epsilon \eta^{(1)\mu}_i(x, u, \partial u) + \mathcal{O}(\epsilon^2), \\ &\vdots \\ \bar{u^{\mu}_{i_1 i_2 \dots i_k}} = U^{\mu}_{i_1 i_2 \dots i_k}(x, u, \partial u, \dots, \partial^k u; \epsilon) &= u^{\mu}_{i_1 i_2 \dots i_k} + \epsilon \eta^{(k)\mu}_{i_1 i_2 \dots i_k}(x, u, \partial u, \dots, \partial^k u) + \mathcal{O}(\epsilon^2), \end{aligned}$$

with the extended infinitesimals $\eta^{(k)\mu}_{i_1 i_2 \dots i_k}$ *given by*

$$\eta^{(1)\mu}_i = D_i \eta^{\mu} - (D_i \xi^j) u^{\mu}_j \tag{22}$$

and

$$\eta^{(k)\mu}_{i_1 i_2 \dots i_k} = D_{i_k} \eta^{(k-1)\mu}_{i_1 i_2 \dots i_{k-1}} - (D_{i_k} \xi^j) u^{\mu}_{i_1 i_2 \dots i_{k-1} j}, \tag{23}$$

where $i_l = 1, 2, \dots, n$ *for* $l = 1, 2, \dots, k$ *with* $k \geq 2$.

Therefore, the kth-extended infinitesimal generator is given by

$$\begin{aligned} X^{(k)} &= \xi^i(x, u) \frac{\partial}{\partial x_i} + \eta^{\mu}(x, u) \frac{\partial}{\partial u^{\mu}} + \eta^{(1)\mu}_i(x, u, \partial u) \frac{\partial}{\partial u^{\mu}_i} + \cdots \\ &+ \eta^{(k)\mu}_{i_1 i_2 \dots i_k}(x, u, \partial u, \partial^2 u, \dots, \partial^k u) \frac{\partial}{\partial u^{\mu}_{i_1 i_2 \cdots i_k}}, k \geq 1. \end{aligned} \tag{24}$$

Example 3.9. *Consider the general generator of* (5) *described in* (15), *where* $\xi^1, \xi^2, \eta^1, \eta^2$ *and* η^3 *are differentiable functions depending on variables* x, t, E, M, N. *Thus, its 2nd-extended infinitesimal generator is given by*

$$\begin{aligned} X^{(2)} &= \xi^1 \partial_x + \xi^2 \partial_t + \eta^1 \partial_E + \eta^2 \partial_M + \eta^3 \partial_N + \eta^{(1)1}_x \partial_{E_x} + \eta^{(1)2}_x \partial_{M_x} + \eta^{(1)3}_x \partial_{N_x} \\ &+ \eta^{(1)1}_t \partial_{E_t} + \eta^{(1)2}_t \partial_{M_t} + \eta^{(1)3}_t \partial_{N_t} + \eta^{(2)1}_{xx} \partial_{E_{xx}} + \eta^{(2)2}_{xx} \partial_{M_{xx}} + \eta^{(2)3}_{xx} \partial_{N_{xx}} \\ &+ \eta^{(2)1}_{xt} \partial_{E_{xt}} + \eta^{(2)2}_{xt} \partial_{M_{xt}} + \eta^{(2)3}_{xt} \partial_{N_{xt}} + \eta^{(2)1}_{tt} \partial_{E_{tt}} + \eta^{(2)2}_{tt} \partial_{M_{tt}} + \eta^{(2)3}_{tt} \partial_{N_{tt}}. \end{aligned}$$

According to equation (21), the derivative operators D_x and D_t are given by

$$\begin{aligned} D_x &= \partial_x + E_x\partial_E + M_x\partial_M + N_x\partial_N + E_{xx}\partial_{E_x} + M_{xx}\partial_{M_x} + N_{xx}\partial_{N_x} \\ &+ E_{xt}\partial_{E_t} + M_{xt}\partial_{M_t} + N_{xt}\partial_{N_t}, \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + E_t\partial_E + M_t\partial_M + N_t\partial_N + E_{tx}\partial_{E_x} + M_{tx}\partial_{M_x} + N_{tx}\partial_{N_x} \\ &+ E_{tt}\partial_{E_t} + M_{tt}\partial_{M_t} + N_{tt}\partial_{N_t}. \end{aligned}$$

Hence, based on further definitions within definition 3.8, the first extended infinitesimals are presented in detail by following expressions. Considering the second extended infinitesimals, only $\eta_{xx}^{(2)1}$ is shown.

$$\begin{aligned} \eta_x^{(1)1} &= D_x\eta^1 - (D_x\xi^1)E_x - (D_x\xi^2)E_t \\ &= \eta_x^1 + E_x\eta_E^1 + M_x\eta_M^1 + N_x\eta_N^1 - (\xi_x^1 + E_x\xi_E^1 + M_x\xi_M^1 + N_x\xi_N^1)E_x \\ &- (\xi_x^2 + E_x\xi_E^2 + M_x\xi_M^2 + N_x\xi_N^2)E_t \end{aligned}$$

$$\begin{aligned} \eta_x^{(1)2} &= D_x\eta^2 - (D_x\xi^1)M_x - (D_x\xi^2)M_t \\ &= \eta_x^2 + E_x\eta_E^2 + M_x\eta_M^2 + N_x\eta_N^2 - (\xi_x^1 + E_x\xi_E^1 + M_x\xi_M^1 + N_x\xi_N^1)M_x \\ &- (\xi_x^2 + E_x\xi_E^2 + M_x\xi_M^2 + N_x\xi_N^2)M_t \end{aligned}$$

$$\begin{aligned} \eta_x^{(1)3} &= D_x\eta^3 - (D_x\xi^1)N_x - (D_x\xi^2)N_t \\ &= \eta_x^3 + E_x\eta_E^3 + M_x\eta_M^3 + N_x\eta_N^3 - (\xi_x^1 + E_x\xi_E^1 + M_x\xi_M^1 + N_x\xi_N^1)N_x \\ &- (\xi_x^2 + E_x\xi_E^2 + M_x\xi_M^2 + N_x\xi_N^2)N_t \end{aligned}$$

$$\begin{aligned} \eta_t^{(1)1} &= D_t\eta^1 - (D_t\xi^1)E_x - (D_t\xi^2)E_t \\ &= \eta_t^1 + E_t\eta_E^1 + M_t\eta_M^1 + N_t\eta_N^1 - (\xi_t^1 + E_t\xi_E^1 + M_t\xi_M^1 + N_t\xi_N^1)E_x \\ &- (\xi_t^2 + E_t\xi_E^2 + M_t\xi_M^2 + N_t\xi_N^2)E_t \end{aligned}$$

$$\begin{aligned} \eta_t^{(1)2} &= D_t\eta^2 - (D_t\xi^1)M_x - (D_t\xi^2)M_t \\ &= \eta_t^2 + E_t\eta_E^2 + M_t\eta_M^2 + N_t\eta_N^2 - (\xi_t^1 + E_t\xi_E^1 + M_t\xi_M^1 + N_t\xi_N^1)M_x \\ &- (\xi_t^2 + E_t\xi_E^2 + M_t\xi_M^2 + N_t\xi_N^2)M_t \end{aligned}$$

$$\begin{aligned} \eta_t^{(1)3} &= D_t\eta^3 - (D_t\xi^1)N_x - (D_t\xi^2)N_t \\ &= \eta_t^3 + E_t\eta_E^3 + M_t\eta_M^3 + N_t\eta_N^3 - (\xi_t^1 + E_t\xi_E^1 + M_t\xi_M^1 + N_t\xi_N^1)N_x \\ &- (\xi_t^2 + E_t\xi_E^2 + M_t\xi_M^2 + N_t\xi_N^2)N_t \end{aligned}$$

$$
\begin{aligned}
\eta_{xx}^{(2)1} &= D_x\eta_x^{(1)1} - (D_x\xi^1)E_{xx} - (D_x\xi^2)E_{xt} \\
&= D_x(\eta_x^1 + E_x\eta_E^1 + M_x\eta_M^1 + N_x\eta_N^1 - (\xi_x^1 + E_x\xi_E^1 + M_x\xi_M^1 + N_x\xi_N^1)E_x \\
&- (\xi_x^2 + E_x\xi_E^2 + M_x\xi_M^2 + N_x\xi_N^2)E_t) - (\xi_x^1 + E_x\xi_E^1 + M_x\xi_M^1 + N_x\xi_N^1)E_{xx} \\
&- (\xi_x^2 + E_x\xi_E^2 + M_x\xi_M^2 + N_x\xi_N^2)E_{xt} \\
&= \eta_{xx}^1 + E_x\eta_{xE}^1 + M_x\eta_{xM}^1 + N_x\eta_{xN}^1 + E_{xx}\eta_E^1 + \eta_{Ex}^1E_x + E_xE_{xE}\eta_E^1 + \eta_{EE}^1(E_x)^2 \\
&+ M_xE_{xM}\eta_E^1 + M_xE_x\eta_{EM}^1 + N_xE_{xN}\eta_E^1 + N_xE_x\eta_{EN}^1 + E_{xx}\eta_E^1 + M_{xx}\eta_M^1 + M_x\eta_{Mx}^1 \\
&+ E_xM_{xE}\eta_M^1 + E_xM_x\eta_{ME}^1 + M_xM_{xM}\eta_M^1 + (M_x)^2\eta_{MM}^1 + N_xM_{xN}\eta_M^1 + N_xM_x\eta_{MN}^1 \\
&+ M_{xx}\eta_M^1 + N_{xx}\eta_N^1 + N_x\eta_{Nx}^1 + E_xN_{xE}\eta_N^1 + E_xN_x\eta_{NE}^1 + M_xN_{xM}\eta_N^1 + M_xN_x\eta_{NM}^1 \\
&+ N_xN_{xN}\eta_N^1 + (N_x)^2\eta_{NN}^1 + N_{xx}\eta_N^1 - \xi_{xx}^1E_x - \xi_x^1E_{xx} - \xi_{xE}^1(E_x)^2 - E_x\xi_x^1E_{xE} \\
&- M_x\xi_{xM}^1E_x - M_x\xi_x^1E_{xM} - N_x\xi_{xN}^1E_x - N_x\xi_x^1E_{xN} - E_{xx}\xi_x^1 - 2E_xE_{xx}\xi_E^1 \\
&- (E_x)^2\xi_{Ex}^1 - 2(E_x)^2E_{xE}\xi_E^1 - (E_x)^3\xi_{EE}^1 - 2M_xE_xE_{xM}\xi_E^1 - M_x(E_x)^2\xi_{EM}^1 \\
&- 2N_xE_xE_{xN}\xi_E^1 - N_x(E_x)^2\xi_{EN}^1 - 2E_{xx}E_x\xi_E^1 - M_{xx}E_x\xi_M^1 - M_xE_{xx}\xi_M^1 - M_xE_x\xi_{Mx}^1 \\
&- M_{xE}(E_x)^2\xi_M^1 - E_xM_xE_{xE}\xi_M^1 - M_x(E_x)^2\xi_{ME}^1 - M_xM_{xM}E_x\xi_M^1 - (M_x)^2E_{xM}\xi_M^1 \\
&- (M_x)^2E_x\xi_{MM}^1 - N_xM_{xN}E_x\xi_M^1 - N_xM_xE_{xN}\xi_M^1 - N_xM_xE_x\xi_{MN}^1 - E_{xx}M_x\xi_M^1 \\
&- M_{xx}E_x\xi_M^1 - N_{xx}E_x\xi_N^1 - N_xE_{xx}\xi_N^1 - N_xE_x\xi_{Nx}^1N_{xE} - (E_x)^2\xi_N^1 - E_xN_xE_{xE}\xi_N^1 \\
&- N_x(E_x)^2\xi_{NE}^1 - M_xN_{xM}E_x\xi_N^1 - M_xN_xE_{xM}\xi_N^1 - M_xN_xE_x\xi_{NM}^1 - N_xN_{xN}E_x\xi_N^1 \\
&- (N_x)^2E_{xN}\xi_N^1 - (N_x)^2E_x\xi_{NN}^1 - E_{xx}N_x\xi_N^1 - N_{xx}E_x\xi_N^1 - \xi_{xx}^2E_t - \xi_x^2E_{tx} - E_x\xi_{xE}^2E_t \\
&- E_x\xi_x^2E_{tE} - M_x\xi_{xM}^2E_t - M_x\xi_x^2E_{tM} - N_x\xi_{xN}^2E_t - N_x\xi_x^2E_{tN} - E_{xt}\xi_x^2 - E_{xx}E_t\xi_E^2 \\
&- E_xE_{tx}\xi_E^2 - E_xE_t\xi_{Ex}^2 - E_xE_{xE}E_t\xi_E^2 - (E_x)^2E_{tE}\xi_E^2 - (E_x)^2E_t\xi_{EE}^2 - M_xE_{xM}E_t\xi_E^2 \\
&- M_xE_xE_{tM}\xi_E^2 - M_xE_xE_t\xi_{EM}^2 - N_xE_{xN}E_t\xi_E^2 - N_xE_xE_{tN}\xi_E^2 - N_xE_xE_t\xi_{EN}^2 \\
&- E_{xx}E_t\xi_E^2 - E_{xt}E_x\xi_E^2 - M_{xx}E_t\xi_M^2 - M_xE_{tx}\xi_M^2 - M_xE_t\xi_{Mx}^2 - E_xM_{xE}E_t\xi_M^2 \\
&- E_xM_xE_{tE}\xi_M^2 - E_xM_xE_t\xi_{ME}^2 - M_xM_{xM}E_t\xi_M^2 - (M_x)^2E_{tM}\xi_M^2 - (M_x)^2E_t\xi_{MM}^2 \\
&- N_xM_{xN}E_t\xi_M^2 - N_xM_xE_{tN}\xi_M^2 - N_xM_xE_t\xi_{MN}^2 - M_{xx}E_t\xi_M^2 - E_{xt}M_x\xi_M^2 \\
&- N_{xx}E_t\xi_N^2 - N_xE_{tx}\xi_N^2 - N_xE_t\xi_{Nx}^2 - E_xN_{xE}E_t\xi_N^2 - E_xN_xE_{tE}\xi_N^2 - E_xN_xE_t\xi_{NE}^2 \\
&- M_xN_{xM}E_t\xi_N^2 - M_xN_xE_{tM}\xi_N^2 - M_xN_xE_t\xi_{NM}^2 - N_xN_{xN}E_t\xi_N^2 - (N_x)^2E_{tN}\xi_N^2 \\
&- (N_x)^2E_t\xi_{NN}^2 - N_{xx}E_t\xi_N^2 - E_{xt}N_x\xi_N^2 - \xi_x^1E_{xx} + E_x\xi_E^1E_{xx} + M_x\xi_M^1E_{xx} + N_x\xi_N^1E_{xx} \\
&- \xi_x^2E_{xt} + E_x\xi_E^2E_{xt} + M_x\xi_M^2E_{xt} + N_x\xi_N^2E_{xt}
\end{aligned}
$$

In a similar way as shown for $\eta_{xx}^{(2)1}$, we can express $\eta_{xx}^{(2)2}, \eta_{xx}^{(2)3}, \eta_{xt}^{(2)1}, \eta_{xt}^{(2)2}, \eta_{xt}^{(2)3}, \eta_{tt}^{(2)1}, \eta_{tt}^{(2)2}$ and $\eta_{tt}^{(2)3}$.

3.4 INFINITESIMAL CRITERION OF INVARIANCE

Theorem 3.4. *(The infinitesimal criterion of invariance of a partial differential equation). Let*

$$X = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu} \tag{25}$$

be the infinitesimal generator of the Lie group of point transformations of (18).

Also let (24) be the kth-extended infinitesimal generator of (25), considering (22) and (23).

So, the one-parameter Lie group of point transformations (18) is admitted by the partial differential equation $F(x,u,\partial u,\partial^2 u,\dots,\partial^k u)$, where $x = (x_1,x_2,x_3,\dots,x_n)$ and $u = (u^1,u^2,u^3,\dots,u^m)$, i.e., is a point Lie symmetry of $F(x,u,\partial u,\partial^2 u,\dots,\partial^k u)$, if and only if

$$X^{(k)}F(x,u,\partial u,\partial^2 u,\dots,\partial^k u) = 0 \quad \textit{when} \quad F(x,u,\partial u,\partial^2 u,\dots,\partial^k u) = 0.$$

Proof: See [18], page 161. ■

Example 3.10. *Let*

$$\begin{aligned} F_1 &= N_t - D_1(N^p N_{xx} + pN^{p-1}N_x^2) + \rho(N_x E_x + NE_{xx}), \\ F_2 &= E_t + \delta ME, \\ F_3 &= M_t - D_2 M_{xx} - \mu N + \lambda M. \end{aligned}$$

The infinitesimal criterion of invariance for system (5) is $X^{(2)}F_\alpha = 0$ (see [7], page 17) when $F_\alpha = 0, \alpha = 1,2,3$, and where F_α represents the equation α of the system (5) in the format $F_\alpha = 0$.

Example 3.11. *Consider F_1, F_2 and F_3 given by the previous example, where $F_1 = F_2 = F_3 = 0$, and the generator $W = X_1 + cX_2$, $X_1 = \partial_t$, $X_2 = \partial_x$.*

$$\begin{aligned} X_1^{(2)}F_1 &= (c_3\partial_t + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\ &+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\ &+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_1 \\ &= 0. \end{aligned}$$

$$\begin{aligned} X_1^{(2)}F_2 &= (c_3\partial_t + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\ &+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\ &+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_2 \\ &= \eta_t^{(1)1}(1). \\ &= 0. \end{aligned}$$

$$\begin{aligned}
X_1^{(2)}F_3 &= (c_3\partial_t + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\
&+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\
&+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_3 \\
&= \eta_t^{(1)2}(1) + \eta_{xx}^{(2)2}(-D_2) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
X_2^{(2)}F_1 &= (c_2\partial_x + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\
&+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\
&+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_1 \\
&= \eta_x^{(1)1}(\rho N_x) + \eta_x^{(1)3}(-2D_1 p N^{p-1} N_x) + \eta_t^{(1)3}(1) + \eta_{xx}^{(2)1}(\rho N) + \eta_{xx}^{(2)3}(-D_1 N^p) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
X_2^{(2)}F_2 &= (c_2\partial_x + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\
&+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\
&+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_2 \\
&= \eta_t^{(1)1}(1) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
X_2^{(2)}F_3 &= (c_2\partial_x + \eta_x^{(1)1}\partial_{E_x} + \eta_x^{(1)2}\partial_{M_x} + \eta_x^{(1)3}\partial_{N_x} \\
&+ \eta_t^{(1)1}\partial_{E_t} + \eta_t^{(1)2}\partial_{M_t} + \eta_t^{(1)3}\partial_{N_t} + \eta_{xx}^{(2)1}\partial_{E_{xx}} + \eta_{xx}^{(2)2}\partial_{M_{xx}} + \eta_{xx}^{(2)3}\partial_{N_{xx}} \\
&+ \eta_{xt}^{(2)1}\partial_{E_{xt}} + \eta_{xt}^{(2)2}\partial_{M_{xt}} + \eta_{xt}^{(2)3}\partial_{N_{xt}} + \eta_{tt}^{(2)1}\partial_{E_{tt}} + \eta_{tt}^{(2)2}\partial_{M_{tt}} + \eta_{tt}^{(2)3}\partial_{N_{tt}})F_3 \\
&= \eta_t^{(1)2}(1) + \eta_{xx}^{(2)2}(-D_2) \\
&= 0.
\end{aligned}$$

Then, $W^{(2)}(F_1) = W^{(2)}(F_2) = W^{(2)}(F_3) = 0$. Hence, we can affirm that W is a generator of system (5).

When we apply the invariance condition to a set of differential equations, we obtain the determining equations for the coefficients of the generator (25). The determining equations form a set of a homogeneous overdetermined linear system of partial differential equations (*PDE*). Then we are able to determine the infinitesimal generators in an explicit form, which is particularly interesting since through them we can obtain invariants and then find invariant solutions that are special solutions of the equations. How to obtain invariants and invariant solutions is the subject of the coming section.

3.5 INVARIANTS

Definition 3.9. *Consider a system S and a one-parameter Lie group of point transformations associated with it defined as in Definition 3.6. Let the system S written as*

$$F_\alpha(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \tag{26}$$

where each α represents a different equation of system S. Consider as well the generator associated with this group as described in equation (25). So $u = \theta(x)$, which its components are $(u^1, u^2, \cdots, u^m) = (\theta_1(x), \theta_2(x), \ldots, \theta_m(x))$, is an invariant solution of the system S if and only if:

1. *$u^i = \theta_i(x)$ is an invariant surface of generator (25) for each $i = 1, 2, \ldots, m$;*
2. *$u = \theta(x)$ solves (26).*

Definition 3.10. *Consider the situation in Definition 3.9. Then $u^i = \theta_i(x)$ is an invariant surface of generator (25) for each $i = 1, 2, \ldots, m$ if $u = \theta(x)$ satisfies:*

$$X(u^i - \theta_i(x)) = 0, \text{ when } u = \theta(x) \text{ for each } i = 1, 2, \ldots, m. \tag{27}$$

According to [6], invariant solutions can be determined by two procedures: *Invariant Form Method* and Direct Substitution Method. At this present work, we chose the first one to apply into system (5), hence we briefly present the method here. The latter one can be seen in [6, page 333].

3.5.1 Invariant Form Method

We can solve the invariant surface conditions (27) by explicitly solving the corresponding *characteristic equations* for $u = \theta(x)$ given by

$$\frac{dx_1}{\xi^1(x,u)} = \frac{dx_2}{\xi^2(x,u)} = \cdots = \frac{dx_n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \frac{du^2}{\eta^2(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}. \tag{28}$$

Solving the $m + n - 1$ first-order ODE system (28) we obtain $m + n - 1$ independent functions named here as $w_1(x, u), w_2(x, u), \ldots, w_{n-1}(x, u), J_1(x, u), J_2(x, u), \ldots, J_m(x, u)$.

In addition to that, when the Jacobian

$$\frac{\partial(J_1, J_2, \ldots, J_m)}{\partial(u^1, u^2, \ldots, u^m)} \neq 0,$$

the general solution $u = \theta(x)$ of the system of *PDE* (27) is given implicitly by the invariant form

$$J_i(x,u) = \Phi_i(w_1(x,u), w_2(x,u), \ldots, w_{n-1}(x,u)), \tag{29}$$

where Φ_i is an arbitrary differentiable function of $w_1(x,u), w_2(x,u), \ldots, w_{n-1}(x,u)$ for $i = 1, 2, \ldots, m$.

Example 3.12. *Consider the system given in Example 3.10 and the information in Example 3.11.*

According to (28), the characteristic equation associated with generator $X_1 + cX_2$ is given by

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dN}{0} = \frac{dE}{0} = \frac{dM}{0}.$$

In order to search for invariants, we have to solve the $m+n-1 = 3+2-1 = 4$ first-order system given by

$$\begin{cases} \frac{dx}{c} & = & \frac{dt}{1}, \\ \frac{dt}{1} & = & \frac{dN}{0}, \\ \frac{dt}{1} & = & \frac{dE}{0}, \\ \frac{dt}{1} & = & \frac{dM}{0}. \end{cases} \tag{30}$$

From integration applied to system (30), the invariants associated with generator $X_1 + cX_2$ can be set by

$$w = x - ct, J_1 = N, J_2 = E, J_3 = M. \tag{31}$$

Then, the Jacobian

$$\frac{\partial(J_1, J_2, J_3)}{\partial(N, E, M)} = \begin{vmatrix} \frac{\partial J_1}{\partial N} & \frac{\partial J_1}{\partial E} & \frac{\partial J_1}{\partial M} \\ \frac{\partial J_2}{\partial N} & \frac{\partial J_2}{\partial E} & \frac{\partial J_2}{\partial M} \\ \frac{\partial J_3}{\partial N} & \frac{\partial J_3}{\partial E} & \frac{\partial J_3}{\partial M} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

So, using (29) and assume $J_1 = \Phi_1(w), J_2 = \Phi_2(w), J_3 = \Phi_3(w)$, where w is as in (31), we have

$$N = \Phi_1(w), E = \Phi_2(w), M = \Phi_3(w). \tag{32}$$

Thus, considering (31) and (32), the system given in (5) can be rewritten as

$$\begin{cases} -c\Phi_1' & = & D_1(\Phi_1^p \Phi_1'' + p\Phi_1^{p-1}\Phi_1'^2) - \rho(\Phi_1'\Phi_2' + \Phi_1\Phi_2''), \\ c\Phi_2' & = & \delta\Phi_3\Phi_2, \\ -c\Phi_3' & = & D_2\Phi_3'' + \mu\Phi_1 - \lambda\Phi_3. \end{cases} \tag{33}$$

The system of *PDE* (26) has invariant solutions given implicitly by the invariant form (29), which one found by solving a reduced system of differential equations with $n-1$ independent variables $w_1(x,u), w_2(x,u), \ldots, w_{n-1}(x,u)$ and m dependent variables $J_1(x,u), J_2(x,u), \ldots, J_m(x,u)$. For details see [6, page 332].

Example 3.13. *Considering the previous example, we can solve the system given in Examples 3.10 and 3.11 by solving the ODE system (33) with w as the independent variable, which was done in Section 5.1.*

In Figure 14 we summarize the process to find solutions of an *ODE* system using Lie Symmetries.

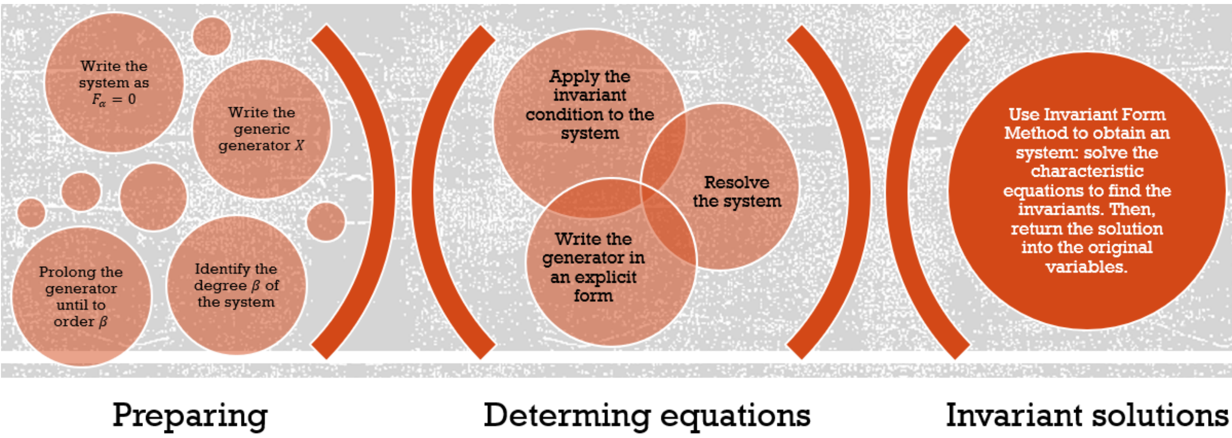


Figure 14: Summarized process to find solutions of an *ODE* system using Lie Symmetries. Source: the author.

4 GROUP CLASSIFICATION

Chapter 3 gives us the tools we need to solve system (5). As we mentioned at the end of previous chapter, we can apply the invariance criterion to the system (5) in order to obtain an overdetermined system of partial differential equations whose coefficients are the infinitesimal generators and their derivatives. The solution leads us to all generators related to system (5).

Applying the invariance condition to a set of partial differential equations usually is a hard and mechanical work. Thus, using a package of software Mathematica [27] called *SYM* developed by [12], we obtained 107 determining equations. Assuming $D_1 D_2 \neq 0$, we reduce the number of these equations to 24 partial differential equations to be solved, whose variables are now the coefficients of the generator (15).

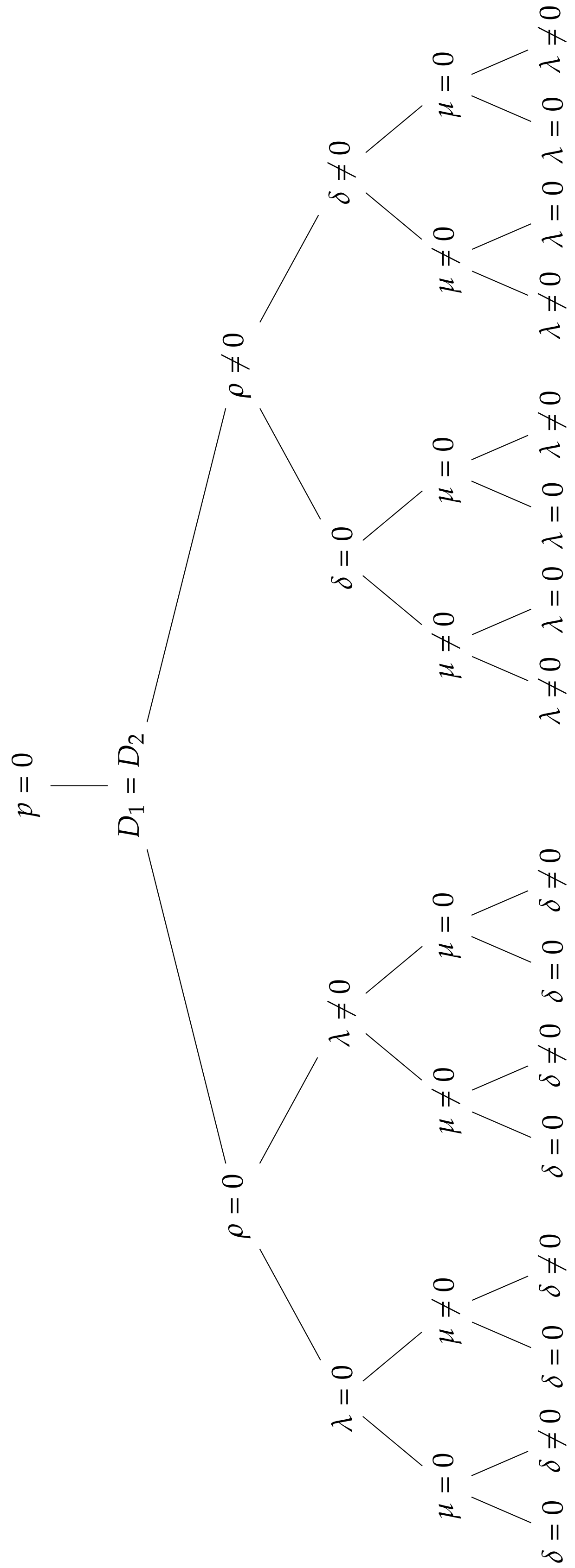
In the solving process we were led to split it into 35 cases based on constraints satisfied by the parameters $D_1, D_2, p, \rho, \mu, \delta, \lambda$. These cases are shown in schemes in the first section of the present chapter.

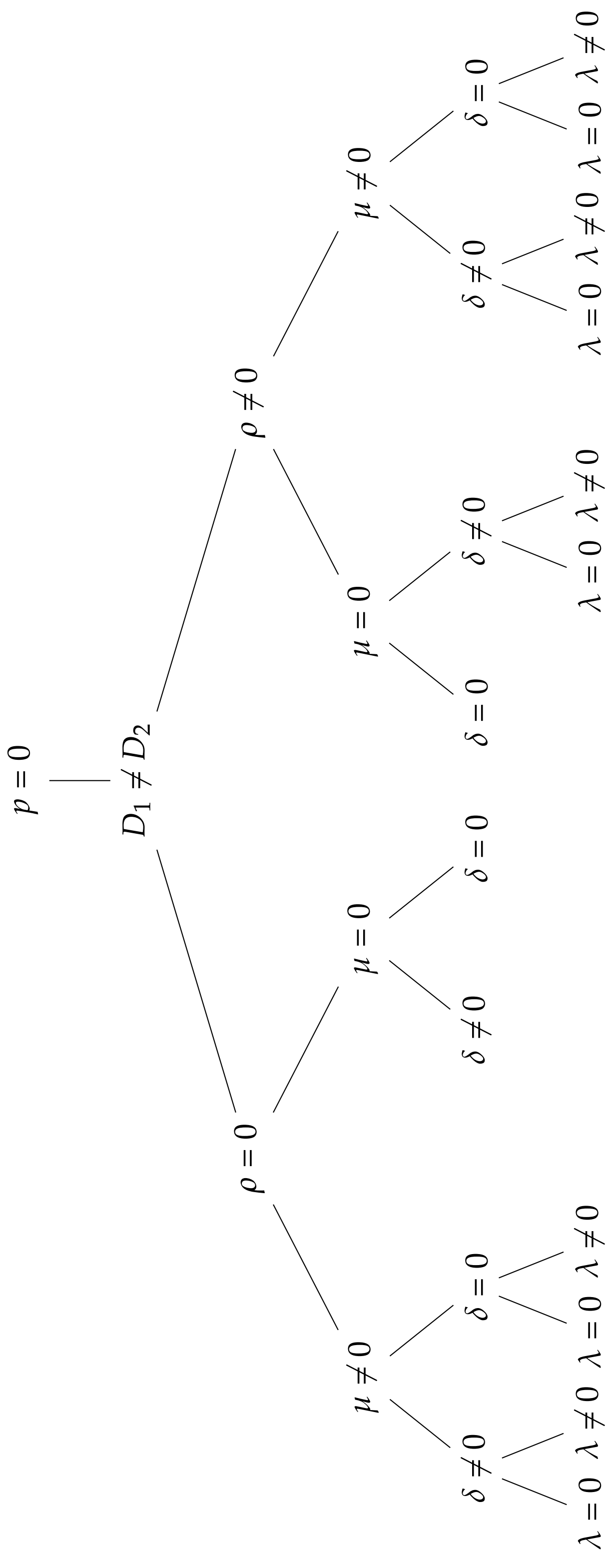
In the second section we present all the generators found, whereas in the last section we solve the already mentioned 24 equations and show how to obtain the findings reported in the second section of this chapter.

A highlight of this chapter is that all propositions and theorems are original and based on the system we want to solve.

4.1 CASES TREES

In this section we show how the cases split according to the parameters. In order to do it we present the cases in a tree format.


p = 0
D1 = D2
ρ ≠ 0
ρ = 0
δ ≠ 0
δ = 0
λ ≠ 0
λ = 0
μ = 0
μ ≠ 0
λ ≠ 0
λ = 0
δ ≠ 0
δ = 0


$p = 0$
$D_1 \neq D_2$
$\rho = 0$
$\rho \neq 0$
$\mu \neq 0$
$\mu = 0$
$\mu = 0$
$\mu \neq 0$
$\delta \neq 0$
$\delta = 0$
$\delta \neq 0$
$\delta = 0$
$\delta = 0$
$\delta \neq 0$
$\delta \neq 0$
$\delta = 0$
$\lambda = 0$
$\lambda \neq 0$
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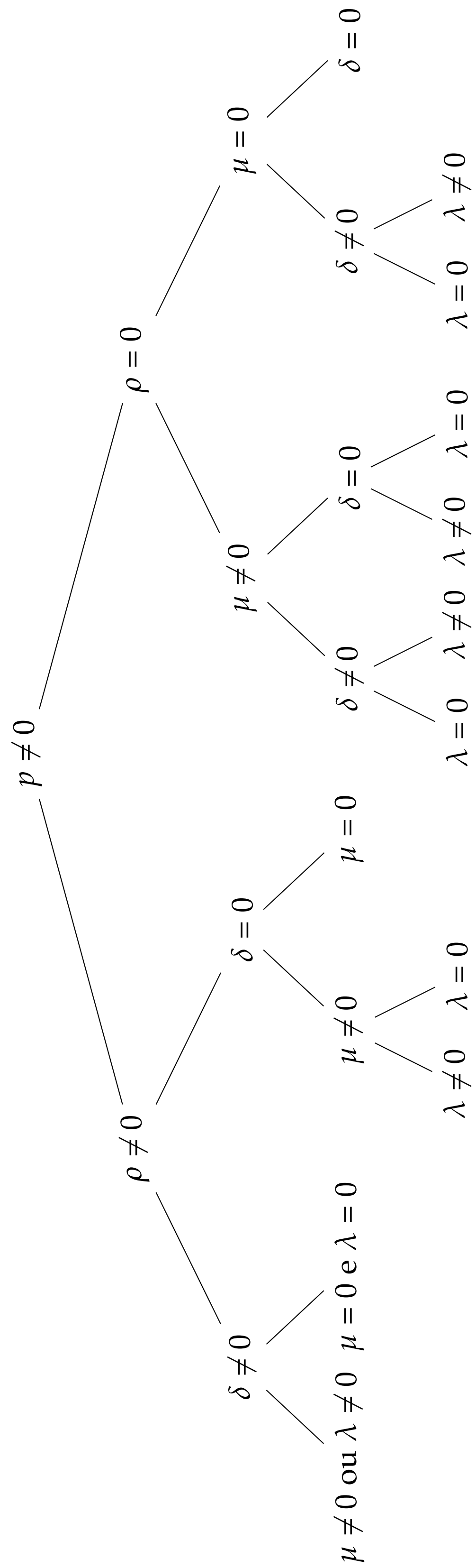
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$


$p \neq 0$
$\rho = 0$
$\rho \neq 0$
$\mu = 0$
$\mu \neq 0$
$\delta = 0$
$\delta \neq 0$
$\lambda = 0$
$\lambda \neq 0$
$\mu = 0$ e $\lambda = 0$
$\mu \neq 0$ ou $\lambda \neq 0$

4.2 INFINITESIMAL GENERATORS

Theorem 4.1. *The infinitesimal generators associated with the system (5) are presented in Tables (2) - (3).*

Proof: See section 4.3. ■

Table 2: Table of generators - $p = 0$ (corresponds to the cases of trees 1 and 2).

	Parameters	Coefficients and generators
1	$\rho = 0, \lambda = 0, \mu = 0,$ $\delta = 0, D_1 = D_2$	$X_1 = \partial_t, X_2 = \partial_x, X_3 = \frac{x}{2}\partial_x + t\partial_t, X_4 = M\partial_M, X_5 = N\partial_M,$ $X_6 = M\partial_N, X_7 = N\partial_N, X_z = z(x,E)\partial_E, Y_h = h(x,t)\partial_M,$ $Z_g = g(x,t)\partial_N,$ $h_t = D_2 h_{xx}, g_t = D_2 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = h(x,E),$ $\eta^2 = c_{11}M + k_1 N + f_3(x,t), \eta^3 = kM + c_{10}N + g_3(x,t),$ $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0,$ $\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$
2	$\rho = 0, \lambda = 0, \mu = 0,$ $\delta \neq 0, D_1 = D_2$	$X_1, X_2, X_6, X_7, X_8 = \frac{x}{2}\partial_x + t\partial_t + E\ln E\partial_E,$ $X_9 = M\partial_M + E\ln E\partial_E, Y_h, Z_g, Y_{fE} = f(x,t)E\partial_E,$ $\delta h = -f_t, h_t = D_2 h_{xx}, g_t = D_2 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2,$ $\eta^1 = E((c_{11} + c_1)\ln E + f_5(x,t)),$ $\eta^2 = c_{11}M + f_3(x,t), \eta^3 = kM + c_{10}N + g_3(x,t),$ $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)), \frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$ $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
3	$\rho = 0, \lambda = 0, \mu \neq 0,$ $\delta = 0, D_1 = D_2$	$X_1, X_2, X_5, X_{10} = M\partial_M - \mu t N\partial_M,$ $X_{11} = M\partial_N + \mu t(M\partial_M - N\partial_N - \mu t N\partial_M),$ $X_{12} = N\partial_N + \mu t N\partial_M, X_{13} = \frac{x}{2}\partial_x + t\partial_t + \mu t N\partial_M, X_z, Y_h,$ $Z_g,$ $h_t - D_2 h_{xx} = \mu g, g_t = D_2 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = h(x,E),$

	Parameters	Coefficients and generators
		$\eta^2=(\mu kt+c_{11})M+(-\mu^2kt^2+\mu t(c_{10}-c_{11}+c_1)+k_1)N$ $+f_3(x,t),\eta^3=kM+(-\mu kt+c_{10})N+g_3(x,t),$ $\frac{\partial}{\partial t}(g_3(x,t))-D_2\frac{\partial^2}{\partial x^2}(g_3(x,t))=0,$ $\frac{\partial}{\partial t}(f_3(x,t))-\mu g_3(x,t)-D_2\frac{\partial^2}{\partial x^2}(f_3(x,t))=0$
4	$\rho=0,\lambda=0,\mu\neq 0,$ $\delta\neq 0,D_1=D_2$	$X_1,X_2,X_{14}=\frac{x}{2}\partial_x+t\partial_t+E\ln E\partial_E-N\partial_N,$ $X_{15}=M\partial_M+E\ln E\partial_E+N\partial_N,Z_g,Y_h,Y_{fE},$ $\delta h=-f_t,h_t-D_2h_{xx}=\mu g,g_t=D_2g_{xx}$ $\xi^1=\frac{c_1}{2}x+c_3,\xi^2=c_1t+c_2,$ $\eta^1=E((c_{11}+c_1)\ln E+f_5(x,t)),$ $\eta^2=c_{11}M+f_3(x,t),\eta^3=(c_{11}-c_1)N+g_3(x,t),$ $\delta f_3(x,t)=-\frac{\partial}{\partial t}(f_5(x,t)),\frac{\partial}{\partial t}(g_3(x,t))-D_2\frac{\partial^2}{\partial x^2}(g_3(x,t))=0,$ $\frac{\partial}{\partial t}(f_3(x,t))-\mu g_3(x,t)-D_2\frac{\partial^2}{\partial x^2}(f_3(x,t))=0$
5	$\rho=0,\lambda\neq 0,\mu\neq 0,$ $\delta=0,D_1=D_2$	$X_1,X_2,$ $X_{16}=\frac{x}{2}\partial_x+t\partial_t+\mu tN\partial_M-\lambda tM\partial_M+(\frac{\mu}{\lambda}-\mu)N\partial_M,$ $X_{17}=\frac{\mu}{\lambda}e^{\lambda t}(M\partial_M-\frac{\mu}{\lambda}N\partial_M)+e^{\lambda t}(M\partial_N-\frac{\mu}{\lambda}N\partial_N),$ $X_{18}=M\partial_M-\frac{\mu}{\lambda}N\partial_M,X_{19}=N\partial_N+\frac{\mu}{\lambda}N\partial_M,$ $X_{20}=e^{-\lambda t}N\partial_M,X_z,Y_h,Z_g,$ $\lambda h+h_t-D_2h_{xx}=\mu g,g_t=D_2g_{xx}$ $\xi^1=\frac{c_1}{2}x+c_3,\xi^2=c_1t+c_2,\eta^1=h(x,E),$ $\eta^2=\left(\frac{\mu k}{\lambda}e^{\lambda t}-c_1\lambda t+c_6\right)M+f_3(x,t)$ $+\left(\frac{\mu}{\lambda}\left(\frac{-\mu k}{\lambda}e^{\lambda t}+c_5-c_6+c_1+c_1\lambda t\right)-\mu c_1+\frac{k_1}{e^{\lambda t}}\right)N,$ $\eta^3=(ke^{\lambda t})M+\left(\frac{-\mu k}{\lambda}e^{\lambda t}+c_5\right)N+g_3(x,t),$ $\frac{\partial}{\partial t}(g_3(x,t))-D_2\frac{\partial^2}{\partial x^2}(g_3(x,t))=0,$ $\frac{\partial}{\partial t}(f_3(x,t))+\lambda f_3(x,t)-\mu g_3(x,t)-D_2\frac{\partial^2}{\partial x^2}(f_3(x,t))=0$
6	$\rho=0,\lambda\neq 0,\mu\neq 0,\delta\neq 0$	$X_1,X_2,X_{15},Y_h,Y_{fE},Z_g,$ $\lambda h+h_t-D_2h_{xx}=\mu g,\delta h=-f_t,g_t=D_2g_{xx}$ $\xi^1=c_3,\xi^2=c_2,\eta^1=E(c_6\ln E+f_5(x,t)),$ $\eta^2=c_6M+f_3(x,t),\eta^3=c_6N+g_3(x,t),$ $\delta f_3(x,t)=-\frac{\partial}{\partial t}(f_5(x,t)),\frac{\partial}{\partial t}(g_3(x,t))-D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))=0,$

	Parameters	Coefficients and generators
		$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
7	$\rho = 0, \lambda \neq 0, \mu = 0,$ $\delta = 0, D_1 = D_2$	$X_1, X_2, X_4, X_7, X_{20}, X_{21} = \frac{x}{2}\partial_x + t\partial_t - \lambda t M\partial_M,$ $X_{22} = e^{\lambda t}M\partial_N, X_z, Y_h, Z_g,$ $\lambda h + h_t - D_2 h_{xx} = 0, g_t = D_2 g_{xx}$ $\xi^1 = \frac{c_1 x}{2} + c_3; \xi^2 = c_1 t + c_2, \eta^1 = h(x,E),$ $\eta^2 = (-\lambda c_1 t + c_6)M + f_3(x,t) + k_1 e^{-\lambda t}N,$ $\eta^3 = ke^{\lambda t}M + g_3(x,t) + c_5 N,$ $\frac{\partial}{\partial t}(g_3(x,t)) = D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)),$ $\lambda f_3(x,t) + \frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
8	$\rho = 0, \lambda \neq 0, \mu = 0,$ $\delta \neq 0, D_1 = D_2$	$X_1, X_2, X_7, X_9, X_{22}, Y_h, Y_{fE}, Z_g,$ $\lambda h + h_t - D_2 h_{xx} = 0, \delta h = -f_t, g_t = D_2 g_{xx}$ $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = E(c_6 \ln E + f_5(x,t)),$ $\eta^2 = c_6 M + f_3(x,t), \eta^3 = (ke^{\lambda t})M + c_5 N + g_3(x,t),$ $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)), \frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
9	$\rho \neq 0, \delta = 0, \mu \neq 0, \lambda \neq 0$	$X_1, X_2, X_{23} = M\partial_M + N\partial_N, X_{24} = \partial_E, Y_h,$ $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = c_4, \eta^2 = c_6 M + f_3(x,t), \eta^3 = c_6 N,$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
10	$\rho \neq 0, \delta = 0, \mu \neq 0,$ $\lambda = 0, D_1 = D_2$	$X_1, X_2, X_{23}, X_{24}, X_{25} = \frac{x}{2}\partial_x + t\partial_t - N\partial_N, Y_h,$ $h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = c_4, \eta^2 = c_6 M + f_3(x,t),$ $\eta^3 = (c_6 - c_1)N,$ $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
11	$\rho \neq 0, \delta = 0, \mu = 0,$ $\lambda = 0, D_1 = D_2$	$X_1, X_2, X_3, X_4, X_7, X_{24}, X_{26} = e^{\frac{\rho E}{D_1}}\partial_N, Y_h,$ $h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = c_4, \eta^2 = c_6 M + f_3(x,t),$ $\eta^3 = c_7 N + ke^{\rho E/D_1},$

	Parameters	Coefficients and generators
		$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
12	$\rho \neq 0, \delta = 0, \mu = 0,$ $\lambda \neq 0, D_1 = D_2$	$X_1, X_2, X_4, X_7, X_{24}, X_{26}, Y_h,$ $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = c_4, \eta^2 = c_6 M + f_3(x,t),$ $\eta^3 = c_7 N + k e^{\rho E/D_1},$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
13	$\rho \neq 0, \delta \neq 0, \mu \neq 0, \lambda \neq 0$	X_1, X_2 $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = 0, \eta^2 = 0, \eta^3 = 0$
14	$\rho \neq 0, \delta \neq 0, \mu \neq 0, \lambda = 0$	$X_1, X_2, X_{27} = \frac{x}{2}\partial_x + t\partial_t - 2N\partial_N - M\partial_M$ $\xi^1 = \frac{c_1 x}{2} + c_3, \xi^2 = c_1 t + c_2, \eta^1 = 0, \eta^2 = -c_1 M,$ $\eta^3 = -2c_1 N$
15	$\rho \neq 0, \delta \neq 0, \mu = 0, \lambda = 0$	$X_1, X_2, X_7, X_{28} = \frac{x}{2}\partial_x + t\partial_t - M\partial_M$ $\xi^1 = \frac{c_1 x}{2} + c_3, \xi^2 = c_1 t + c_2, \eta^1 = 0, \eta^2 = -c_1 M, \eta^3 = c_7 N$
16	$\rho \neq 0, \delta \neq 0, \mu = 0, \lambda \neq 0$	X_1, X_2, X_7 $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = 0, \eta^2 = 0, \eta^3 = c_7 N$
17	$\rho = 0, \mu \neq 0, \delta \neq 0,$ $\lambda = 0, D_1 \neq D_2$	$X_1, X_2, X_{15}, X_{29} = \frac{x}{2}\partial_x + t\partial_t + E\ln E\partial_E - N\partial_N, Y_h, Y_{fE},$ $Z_g,$ $h_t - D_2 h_{xx} = \mu g, \delta h = -f_t, g_t = D_1 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = E((c_6 + c_1)\ln E + f_5(x,t)),$ $\eta^2 = c_6 M + f_3(x,t), \eta^3 = (c_6 - c_1)N + g_3(x,t),$ $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0,$ $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)), \frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$
18	$\rho = 0, \mu \neq 0, \delta = 0,$ $\lambda = 0, D_1 \neq D_2$	$X_1, X_2, X_{23}, X_{30} = \frac{x}{2}\partial_x + t\partial_t - N\partial_N, X_z, Y_h, Z_g,$ $h_t - D_2 h_{xx} = \mu g, g_t = D_1 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = h(x,E),$ $\eta^2 = c_6 M + f_3(x,t),$

	Parameters	Coefficients and generators
		$\eta^3 = (c_6 - c_1)N + g_3(x,t)$, $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0$, $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$
19	$\rho = 0, \mu \neq 0, \delta = 0$ $\lambda \neq 0, D_1 \neq D_2$	$X_1, X_2, X_{23}, X_z, Y_h, Z_g$, $\lambda h + h_t - D_2 h_{xx} = \mu g, g_t = D_1 g_{xx}$ $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = h(x,E), \eta^2 = c_6 M + f_3(x,t)$, $\eta^3 = c_6 N + g_3(x,t)$, $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0$, $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$
20	$\rho = 0, \mu = 0, \delta \neq 0$, $\forall\lambda, D_1 \neq D_2$	$X_1, X_2, X_7, X_8, X_9, Y_h, Z_g$, $\lambda h + h_t - D_2 h_{xx} = 0, \delta h = -f_t, g_t = D_1 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = E((c_6 + c_1)\ln E + f_5(x,t))$, $\eta^2 = c_6 M + f_3(x,t), \eta^3 = c_7 N + g_3(x,t)$, $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$, $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)), \frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$
21	$\rho = 0, \mu = 0, \delta = 0$, $\forall\lambda, D_1 \neq D_2$	$X_1, X_2, X_4, X_7, X_{21}, X_z, Y_h, Z_g$, $\lambda h + h_t - D_2 h_{xx} = 0, g_t = D_1 g_{xx}$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = h(x,E)$, $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t), \eta^3 = c_7 N + g_3(x,t)$, $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$, $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$
22	$\rho \neq 0, \mu = 0, \delta = 0$, $\forall\lambda, D_1 \neq D_2$	$X_1, X_2, X_4, X_7, X_{21}, X_{24}, X_{26}, Y_h, \lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = c_4$, $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$, $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t))$, $\eta^3 = c_5 N + k e^{\rho E/D_1}$
23	$\rho \neq 0, \mu \neq 0, \delta = 0, \lambda = 0$	$X_1, X_2, X_{23}, X_{24}, X_{31} = \frac{x}{2}\partial_x + t\partial_t + M\partial_M, Y_h$, $h_t - D_2 h_{xx} = 0$

	Parameters	Coefficients and generators
		$\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = c_4,$ $\eta^2 = (c_5 + c_1)M + f_3(x,t), \eta^3 = c_5 N,$ $\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$

Table 3: Table of generators - $p \neq 0$ (corresponds to the cases of tree 3).

	Parameters	Coefficients and generators
24	$\rho \neq 0, \delta \neq 0, (\mu \neq 0$ ou $\lambda \neq 0)$	X_1, X_2 $\xi^1 = c_3; \xi^2 = c_2; \eta^1 = 0; \eta^2 = 0; \eta^3 = 0$
25	$\rho \neq 0, \delta \neq 0, (\mu = 0$ e $\lambda = 0)$	X_1, X_2, X_{28} $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = 0, \eta^2 = -c_1 M,$ $\eta^3 = 0$
26	$\rho \neq 0, \delta = 0, \mu = 0, \forall\lambda$	$X_1, X_2, X_4, X_{21}, X_{24}, Y_h,$ $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3; \xi^2 = c_1 t + c_2;\ \eta^1 = c_4;$ $\eta^2 = (c_5 - \lambda c_1 t)M + f_3(x,t);$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)); \eta^3 = 0$
27	$\rho \neq 0, \delta = 0, \mu \neq 0, \lambda \neq 0$	$X_1, X_2, X_{24}, Y_h,$ $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = c_3, \xi^2 = c_2;\ \eta^1 = c_4;$ $\eta^2 = f_3(x,t),$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)),$ $\eta^3 = 0$
28	$\rho \neq 0, \delta = 0, \mu \neq 0, \lambda = 0$	$X_1, X_2, X_{24}, X_{31}, Y_h,$ $h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2, \eta^1 = c_4,$ $\eta^2 = c_1 M + f_3(x,t), \frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)),$

	Parameters	Coefficients and generators
		$\eta^3 = 0$
29	$\rho = 0, \mu \neq 0, \delta \neq 0, \lambda = 0$	$X_1, X_2, X_{32} = \frac{x}{2}\partial_x + t\partial_t + 2E\ln E\partial_E + M\partial_M, Y_h, Y_{fE},$ $h_t - D_2 h_{xx} = 0, \delta h = -f_t$ $\xi^1 = \frac{c_1}{2}x + c_3, \xi^2 = c_1 t + c_2,$ $\eta^1 = (2c_1 \ln E + f_5(x,t))E,$ $\eta^2 = c_1 M + f_3(x,t),$ $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$ $\frac{\partial}{\partial t}(f_3(x,t)) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)),$ $\eta^3 = 0$
30	$\rho = 0, \mu \neq 0, \delta \neq 0, \lambda \neq 0$	$X_1, X_2, Y_h, Y_{fE},$ $\lambda h + h_t - D_2 h_{xx} = 0, \delta h = -f_t$ $\xi^1 = c_3, \xi^2 = c_2, \eta^1 = f_5(x,t)E,$ $\eta^2 = f_3(x,t), \delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)), \eta^3 = 0$
31	$\rho = 0, \mu \neq 0, \delta = 0, \lambda = 0$	$X_1, X_2, X_{31}, Y_h, X_z,$ $h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1}{2}x + c_3; \xi^2 = c_1 t + c_2; \eta^1 = h(x,E);$ $\eta^2 = c_1 M + f_3(x,t), \frac{\partial}{\partial t}(f_3(x,t)) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t));$ $\eta^3 = 0$
32	$\rho = 0, \mu \neq 0, \delta = 0, \lambda \neq 0$	$X_1, X_2, Y_h, X_z,$ $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = c_3; \xi^2 = c_2; \eta^1 = h(x,E);$ $\eta^2 = f_3(x,t),$ $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t));$ $\eta^3 = 0$
33	$\rho = 0, \mu = 0, \delta \neq 0, \lambda = 0$	$X_1, X_2, X_8, X_9, Y_h, Y_{fE},$ $h_t - D_2 h_{xx} = 0, \delta h = -f_t$ $\xi^1 = \frac{c_1 x}{2} + c_3; \xi^2 = c_1 t + c_2;$

	Parameters	Coefficients and generators
		$\eta^1 = E((c_5 + c_1)\ln E + f_5(x,t))$, $\eta^2 = c_5 M + f_3(x,t)$, $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$; $\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$; $\eta^3 = 0$
34	$\rho = 0, \mu = 0, \delta \neq 0, \lambda \neq 0$	$X_1, X_2, X_9, Y_h, Y_{fE}$, $\lambda h + h_t - D_2 h_{xx} = 0, \delta h = -f_t$ $\xi^1 = c_3$; $\xi^2 = c_2$; $\eta^1 = E(c_5 \ln E + f_5(x,t))$, $\eta^2 = c_5 M + f_3(x,t)$, $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$; $\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)) - \lambda f_3(x,t)$; $\eta^3 = 0$
35	$\rho = 0, \mu = 0, \delta = 0, \forall\lambda$	$X_1, X_2, X_4, X_{21}, X_z, Y_h$, $\lambda h + h_t - D_2 h_{xx} = 0$ $\xi^1 = \frac{c_1 x}{2} + c_3$; $\xi^2 = c_1 t + c_2$; $\eta^1 = h(x,E)$; $\eta^2 = (-c_1 \lambda t + c_5)M + f_3(x,t)$, $\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)) - \lambda f_3(x,t)$; $\eta^3 = 0$

4.3 THE DETERMINING EQUATIONS

Using the software Mathematica [27] and its package SYM [12] to obtain the determining equations, assuming $D_1 D_2 \neq 0$, we can reduce the determining equations for the system (5) as following:

$$\xi^1_N = \xi^1_E = \xi^1_M = \xi^1_t = 0, \tag{34}$$

$$\xi^2_x = \xi^2_E = \xi^2_M = \xi^2_N = 0, \tag{35}$$

$$\xi^2_t - 2\xi^1_x = 0, \tag{36}$$

$$\eta^1_N = \eta^1_M = 0, \tag{37}$$

$$\eta^2_{xE} = \eta^2_{EE} = \eta^2_{EM} = \eta^2_{MM} = \eta^2_{xN} = \eta^2_{MN} = 0, \tag{38}$$

$$\eta^3_{xM} = \eta^3_{MM} = 0, \tag{39}$$

$$p\eta^3 = 0, \tag{40}$$

$$(D_2 - D_1 N^p)\eta^3_M = 0, \tag{41}$$

$$2\eta^2_{xM} - \xi^1_{xx} = 0, \tag{42}$$

$$(D_2 - D_1 N^p)\eta^2_N = 0, \tag{43}$$

$$\rho N \eta^2_N + D_2 \eta^2_E = 0, \tag{44}$$

$$\rho \eta^2_N + 2D_2 \eta^2_{EN} = 0, \tag{45}$$

$$pD_1 N^p \eta^2_N - D_2 N \eta^2_{NN} = 0, \tag{46}$$

$$\rho(\eta^3_E + N\eta^1_{EE}) - D_1 N^p \eta^3_{EE} = 0, \tag{47}$$

$$\eta^3_t + \rho N \eta^1_{xx} - \delta E M \eta^3_E - D_1 N^p \eta^3_{xx} + (\mu N - \lambda M)\eta^3_M = 0, \tag{48}$$

$$\rho(\eta^3_x + 2N\eta^1_{xE}) - 2D_1 N^p \eta^3_{xE} - N^p \xi^1_{xx} = 0, \tag{49}$$

$$\rho \eta^3_M - 2D_1 N^p \eta^3_{EM} = 0, \tag{50}$$

$$\rho(\eta^3 - N\eta^3_N + N\eta^1_E) - D_1 N^p \eta^3_E = 0, \tag{51}$$

$$p\eta^3_M + N\eta^3_{MN} = 0, \tag{52}$$

$$\eta^1_t + \delta(M\eta^1 + E\eta^2 - EM\eta^1_E + EM\xi^2_t) = 0, \tag{53}$$

$$\rho N\eta^1_x - 2D_1N^p(p\eta^3_x + N\eta^3_{xN}) + D_1N^{p+1}\xi^1_{xx} = 0, \tag{54}$$

$$\rho N\eta^1_E - 2D_1N^p(p\eta^3_E - N\eta^3_{EN}) = 0, \tag{55}$$

$$\eta^2_t + \lambda\eta^2 - \mu\eta^3 - \lambda M\eta^2_M + \mu N\eta^2_M - \delta EM\eta^2_E + \lambda M\xi^2_t - \mu N\xi^2_t - D_2\eta^2_{xx} = 0, \tag{56}$$

$$N^p(p-1)p\eta^3 + N^{p+1}(p\eta^3_N + N\eta^3_{NN}) = 0. \tag{57}$$

Proposition 4.1. *If ξ^1 and ξ^2 satisfy (34), (35) and (36), then*

$$\xi^1 = \frac{c_1 x}{2} + c_3, \tag{58}$$

$$\xi^2 = c_1 t + c_2,$$

where c_1, c_2 and c_3 are arbitrary constants.

Proof: Equations (34) and (35) imply, respectively, that $\xi^1 = \xi^1(x)$ and $\xi^2 = \xi^2(t)$. In addition, taking (36) into account, we obtain the result. ∎

Proposition 4.2. *If η^1 satisfies (37), η^2 satisfies (38) and (42), and ξ^1 satisfies (58), then $\eta^1 = h(x,t,E)$ and $\eta^2 = f_1(t)M + f_2(t,N)E + f_3(x,t) + f_4(t,N)$, where h, f_1, f_2, f_3 and f_4 are smooth functions.*

Proof: From equations (37), we have $\eta^1 = h(x,t,E)$, where h is an arbitrary smooth function. Since $\xi^1 = \frac{c_1 x}{2} + c_3$, then $\xi^1_{xx} = 0$ and from (42) we obtain $\eta^2_{xM} = 0$. From equation (38) and the fact that $\eta^2_{xM} = 0$, we conclude that

$$\eta^2 = f_1(t)M + f_2(t,N)E + f_3(x,t) + f_4(t,N), \tag{59}$$

where f_1, f_2, f_3 and f_4 are arbitrary smooth functions. ∎

Proposition 4.3. *If η^3 satisfies (39), then*

$$\eta^3 = g(t,E,N)M + g_1(x,t,E,N). \tag{60}$$

Proof: The second equation in (39) implies that η^3 is linear in M whereas the first equation says that coefficient of M does not depend on x and this proves the result. ∎

4.3.1 The case p nonzero

Through this subsection we will assume $p \neq 0$.

Proposition 4.4. *If η^3 satisfies equations (40) and (60), then $\eta^3 = 0$. In particular, $g = g_1 = 0$.*

Proof: It follows immediately from (40). ∎

Proposition 4.5. *If η^2 satisfies equations (43) and (44), then $\eta^2 = f_1(t)M + f_3(x,t)$.*

Proof: From (43) we obtain $\eta^2_N = 0$. By (59), this condition is equivalent to $f_{2_N} = 0$ and $f_{4_N} = 0$. For simplicity $f_4(t)$ can be merged with $f_3(x,t)$. On the other hand, (44) implies $\eta^2_E = 0$, which gives $f_2 = 0$. These conditions give the result. ∎

Until this point, the system of determining equations can be written as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2, \tag{61}$$

$$\eta^1 = h(x,t,E), \tag{62}$$

$$\eta^2 = f_1(t)M + f_3(x,t), \tag{63}$$

$$\eta^3 = 0, \tag{64}$$

$$\eta^1_t + \delta(M\eta^1 + E\eta^2 - EM\eta^1_E + c_1 EM) = 0, \tag{65}$$

$$\rho N \eta^1_x = 0, \tag{66}$$

$$\rho N \eta^1_E = 0, \tag{67}$$

$$\eta^2_t + \lambda\eta^2 - \lambda M f_1(t) + \mu N f_1(t) + c_1\lambda M - c_1\mu N - D_2\eta^2_{xx} = 0. \tag{68}$$

The case $\rho \neq 0$*:*

In addition to the condition $p \neq 0$, we add the condition $\rho \neq 0$.

Proposition 4.6. *If* η^1 *satisfies equations* (62), (66) *and* (67), *then* $\eta^1_x = 0$ *and* $\eta^1_E = 0$. *In particular,* $h = h(t)$.

Proof: It is immediate from equations (62), (66) and (67). ■

Deriving equation (65) with respect to E and by Proposition 4.6, we have:

$$\delta(\eta^2 + c_1 M) = 0. \tag{69}$$

Equation (69) suggests that we can divide this case into two: $\delta = 0$ and $\delta \neq 0$.

Proposition 4.7. *If* $\delta = 0$*, Proposition* 4.6 *is satisfied and* η^1 *satisfies equation* (65)*, then* $\eta^1 = c_4$*, where* c_4 *is a constant.*

Proof: By Proposition 4.6 we have $\eta^1 = h(t)$. From (65) we have $h'(t) = 0$ and we obtain the desired result. ■

Proposition 4.8. *If* $\delta = 0$ *and* η^2 *satisfies equations* (63) *and* (68)*, then* $f_1 = c_5 - \lambda c_1 t$*,* $\mu(f_1(t) - c_1) = 0$ *and*

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Proof: Replacing equation (63) into (68), we have the identity

$$M(f_1'(t) + c_1\lambda) + N(\mu f_1 - \mu c_1) + \tfrac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2 \tfrac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Then,

$$f_1'(t) + \lambda c_1 = 0 \Rightarrow f_1(t) = c_5 - \lambda c_1 t,$$

$$\mu(f_1(t) - c_1) = 0, \tag{70}$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

■

Proposition 4.9. *If $\delta = 0$, $\mu \neq 0$, $\lambda \neq 0$ and Proposition 4.8 is satisfied, then $f_1 = 0$.*

Proof: If $\mu \neq 0$, equation (70) gives $\lambda c_1 = 0$ and $c_5 = c_1$. Since $\lambda \neq 0$, $c_1 = 0$ and we have the result. ■

Proposition 4.10. *If $\delta = 0$, $\mu \neq 0$, $\lambda = 0$ and Proposition 4.8 is satisfied, then $f_1 = c_1$.*

Proof: If $\mu \neq 0$, equation (70) gives $\lambda c_1 = 0$ and $c_5 = c_1$. ■

Therefore, as long as $p \neq 0$, $\rho \neq 0$, $\delta = 0$ and $\mu = 0$, the solution of the determining equations is:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = (c_5 - \lambda c_1 t)M + f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Considering $p \neq 0$, $\rho \neq 0$, $\delta = 0$, $\mu \neq 0$ and $\lambda \neq 0$, the solution is:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Whilst for $p \neq 0$, $\rho \neq 0$, $\delta = 0$, $\mu \neq 0$ and $\lambda = 0$, we have the solution:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = c_1 M + f_3(x,t),$$

$$\eta^3 = 0,$$

with

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Proposition 4.11. *If $\delta \neq 0$, ξ^1 and ξ^2 satisfy equations (61), η^1 and η^2 satisfy equations (65), (68) and (69), and Proposition 4.6 is satisfied, then $\eta^2 = -c_1 M$ and $\eta^1 = 0$. In particular, if $\lambda \neq 0$ or $\mu \neq 0$, then $\eta^2 = 0$, $\xi^1 = c_3$ and $\xi^2 = c_2$.*

Proof: Condition (69) implies $\eta^2 = -c_1 M$ when $\delta \neq 0$. Substituting this into (68) we have $\lambda c_1 M = 0$ and hence $-2\mu c_1 N + \lambda c_1 M = 0$, which implies $\mu c_1 = 0$ and $\lambda c_1 = 0$. If $\lambda \neq 0$ or $\mu \neq 0$, then $c_1 = 0$ which implies $\eta^2 = 0$, $\xi^1 = c_3$ and $\xi^2 = c_2$.

Equation (65) jointly with Proposition 4.6, $\eta^2 = -c_1 M$ and the condition $\delta \neq 0$, yield $\eta^1 = 0$. ■

Then, for $p \neq 0$, $\rho \neq 0$, $\delta \neq 0$, $\lambda \neq 0$ and $\forall \mu$:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = 0.$$

If $p \neq 0$, $\rho \neq 0$, $\delta \neq 0$, $\forall \lambda$ and $\mu \neq 0$, we have:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = 0.$$

And the solution of the determining equations for $p \neq 0$, $\rho \neq 0$, $\delta \neq 0$, $\lambda = 0$ and $\mu = 0$ is given by

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = 0, \quad \eta^2 = -c_1 M, \quad \eta^3 = 0.$$

The case *$\rho = 0$:*

In addition to the condition $p \neq 0$, we add the condition $\rho = 0$.

Proposition 4.12. *If η^2 satisfies equations (63) and (68), then $f_1 = c_5 - \lambda c_1 t$, $\mu(f_1(t) - c_1) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$.*

Proof: From equation (63) we know $\eta^2_{tM} = f_1'(t)$. So, deriving equation (68) with respect to M and N, respectively, we have:

$$f_1'(t) + \lambda c_1 = 0,$$

$$\mu(f_1(t) - c_1) = 0. \tag{71}$$

Then, we have that $f_1'(t) = -\lambda c_1$, which implies $f_1(t) = c_5 - \lambda c_1 t$, and $\mu(f_1(t) - c_1) = 0$. Using both information into (68), we conclude $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$. ■

Proposition 4.13. *If $\delta \neq 0$, ξ^1 and ξ^2 satisfy equations (61), η^1 satisfies equation (62), η^1 and η^2 satisfy equation (65), and Proposition 4.12 is satisfied, then $\eta^1 = ((c_5 + c_1)\ln E + f_5(x,t))E$, where $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$, and $\eta^2 = c_5 M + f_3(x,t)$. In particular, if $\lambda \neq 0$, then $\xi^1 = c_3$, $\xi^2 = c_2$ and $\eta^1 = (c_5 \ln E + f_5(x,t))E$.*

Proof: Since $\eta^1 = h(x,t,E)$, $\eta^2 = f_1(t)M + f_3(x,t) = (c_5 - \lambda c_1 t)M + f_3(x,t)$ and $\delta \neq 0$, considering the independent variables, from equation (65) we have:

$$\eta^1 + E(c_5 + c_1 - \eta^1_E - \lambda c_1 t) = 0, \tag{72}$$

$$\eta^1_t + \delta E f_3(x,t) = 0. \tag{73}$$

From equation (73), we obtain $\eta^1_t = -\delta E f_3(x,t)$. Comparing that with the derivative of equation (72) with respect to t, obtain $\lambda c_1 = 0$. Therefore, resolving equation (72) by integrating factor, we can affirm

$$\eta^1 = ((c_5 + c_1)\ln E + f_5(x,t))E. \tag{74}$$

From equations (73) and (74), we have $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$.

In particular, if $\lambda \neq 0$, then $c_1 = 0$, which implies $\xi^1 = c_3$, $\xi^2 = c_2$ and $\eta^1 = (c_5 \ln E + f_5(x,t))E$. ■

Proposition 4.14. *If $\delta = 0$, then $\eta^1 = h(x,E)$.*

Proof: It comes directly from equation (65). ■

Proposition 4.15. *If $\mu \neq 0$, Proposition 4.12 is satisfied and ξ^1 and ξ^2 satisfy equations (61), then $f_1 = c_1$, i.e., $c_5 = c_1$ and $\lambda c_1 = 0$. In particular, if $\lambda \neq 0$, then $\xi^1 = c_3$, $\xi^2 = c_2$ and $\eta^2 = f_3(x,t)$.*

Proof: If $\mu \neq 0$, equation (71) gives $f_1 = c_1$. Since $f_1(t) = c_5 - \lambda c_1 t$, then $c_5 = c_1$ and $\lambda c_1 = 0$, which implies $\eta^2 = c_1 M + f_3(x,t)$. In particular, if $\lambda \neq 0$, then $c_1 = 0$, i.e., $\xi^1 = c_3$ and $\xi^2 = c_2$ come directly from equations (61), and $\eta^2 = f_3(x,t)$. ■

Hence, when $p \neq 0, \rho = 0, \delta \neq 0, \mu \neq 0$ and $\lambda \neq 0$, we can rewrite the system as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = f_5(x,t)E,$$

$$\eta^2 = f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Considering $p \neq 0, \rho = 0, \delta \neq 0, \mu \neq 0$ and $\lambda = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = (2c_1 \ln E + f_5(x,t))E,$$

$$\eta^2 = c_1 M + f_3(x,t),$$

$$\eta^3 = 0,$$

with

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Since $p \neq 0, \rho = 0, \delta = 0, \mu \neq 0$ and $\lambda \neq 0$, the system is rewritten as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Setting $p \neq 0, \rho = 0, \delta = 0, \mu \neq 0$ and $\lambda = 0$, we obtain the system :

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = c_1 M + f_3(x,t),$$

$$\eta^3 = 0,$$

with

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

On the other hand, for $p \neq 0, \rho = 0, \delta \neq 0, \mu = 0$ and $\lambda = 0$, the system becomes:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = E((c_5 + c_1)\ln E + f_5(x,t)),$$

$$\eta^2 = c_5 M + f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Considering $p \neq 0, \rho = 0, \delta \neq 0, \mu = 0$ and $\lambda \neq 0$, we can rewrite the system as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = E(c_5 \ln E + f_5(x,t)),$$

$$\eta^2 = c_5 M + f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \lambda f_3(x,t).$$

For $p \neq 0, \rho = 0, \delta = 0$ and $\mu = 0$, we obtain the system :

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = (c_5 - c_1\lambda t)M + f_3(x,t),$$

$$\eta^3 = 0,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

4.3.2 The case $p = 0$

Through this subsection we assume $p = 0$.

Proposition 4.16. *If η^2 satisfies equations (44) - (46) and Proposition 4.2 is satisfied, then $\eta^2 = f_1(t)M + f_2(t)E + f_3(x,t) + f_4(t)N$.*

Proof: Replacing $p = 0$ in equation (46), we have $\eta^2_{NN} = 0$. So, deriving equation (44) with respect to N, we obtain $\rho\eta^2_N + \rho N\eta^2_{NN} + D_2\eta^2_{EN} = 0$. Then, we have:

$$\rho\eta^2_N + D_2\eta^2_{EN} = 0. \tag{75}$$

Making equation (45) - equation (75), we have $\eta^2_{EN} = 0$. Taking into account equation (59) within Proposition 4.2 and the results above, we obtain $\eta^2 = f_1(t)M + f_2(t)E + f_3(x,t) + f_4(t)N$. ∎

Proposition 4.17. *If η^1 satisfies equations (37), (51) and (55), η^3 satisfies Proposition 4.3 and equations (50) - (52), (55) and (57), then $\eta^3 = g(t)M + Ng_2(x,t) + g_3(x,t,E)$.*

Proof: Assuming that Proposition 4.3 holds and replacing $p = 0$ into equations (52) and (57), we have $\eta^3_{MN} = 0$ and $\eta^3_{NN} = 0$, respectively, which implies $\eta^3 = g(t,E)M + Ng_2(x,t,E) + g_3(x,t,E)$.

Deriving equation (51) with respect to N, we obtain $\rho\eta^3_N - \rho\eta^3_N - N\rho\eta^3_{NN} + \rho\eta^1_E + N\rho\eta^1_{EN} - D_1\eta^3_{EN} = 0$, and we can replace $\eta^3_{NN} = 0$ into this equation. Besides, we know that $\eta^1_N = 0$ from equations (37). So, we have:

$$\rho\eta^1_E - D_1\eta^3_{EN} = 0. \tag{76}$$

Doing equation (55) - N $\times$ equation (76), we have $\eta^3_{EN} = 0$.

On the other hand, the derivative of equation (51) with respect to M is $\rho\eta^3_M - \rho N\eta^3_{NM} + \rho N\eta^1_{EM} - D_1\eta^3_{EM} = 0$. We can replace $\eta^3_{NM} = 0$ into this last equation obtained and we also know $\eta^1_M = 0$ from equations (37). So, we have:

$$\rho\eta^3_M - D_1\eta^3_{EM} = 0. \tag{77}$$

Doing equation (50) - equation (77), we have that $\eta^3_{EM} = 0$.

Therefore, $\eta^3 = g(t)M + Ng_2(x,t) + g_3(x,t,E)$. ■

Then, the system can be rewritten as:

$$\xi^1 = \frac{c_1 x}{2} + c_3, \quad \xi^2 = c_1 t + c_2, \tag{78}$$

$$\eta^1 = h(x,t,E), \tag{79}$$

$$\eta^2 = f_1(t)M + f_2(t)E + f_3(x,t) + f_4(t)N, \tag{80}$$

$$\eta^3 = g(t)M + g_2(x,t)N + g_3(x,t,E), \tag{81}$$

$$(D_2 - D_1)g(t) = 0, \tag{82}$$

$$(D_2 - D_1)\eta^2_N = 0, \tag{83}$$

$$\rho N\eta^2_N + D_2 f_2(t) = 0, \tag{84}$$

$$\rho\eta^2_N = 0, \tag{85}$$

$$\rho(\eta^3_E + N\eta^1_{EE}) - D_1\eta^3_{EE} = 0, \tag{86}$$

$$\eta^3_t + \rho N\eta^1_{xx} - \delta EM\eta^3_E - D_1\eta^3_{xx} + (\mu N - \lambda M)g(t) = 0, \tag{87}$$

$$\rho(\eta^3_x + 2N\eta^1_{xE}) - 2D_1\eta^3_{xE} = 0, \tag{88}$$

$$\rho\eta^3_M = 0, \tag{89}$$

$$\rho(\eta^3 - N\eta^3_N + N\eta^1_E) - D_1\eta^3_E = 0, \tag{90}$$

$$\eta^1_t + \delta(M\eta^1 + E\eta^2 - EM\eta^1_E + c_1EM) = 0, \tag{91}$$

$$\rho\eta^1_x - 2D_1\eta^3_{xN} = 0, \tag{92}$$

$$\rho\eta^1_E = 0, \tag{93}$$

$$\eta^2_t + \lambda\eta^2 - \mu\eta^3 - \lambda Mf_1(t) + \mu Nf_1(t) - \delta EMf_2(t) + c_1\lambda M - c_1\mu N - D_2\eta^2_{xx} = 0. \tag{94}$$

The case $D_1 \neq D_2$***:***

In addition to the condition $p = 0$, we add the condition $D_1 \neq D_2$.

Proposition 4.18. *If η^2 satisfies equations (80), (83) and (84), η^3 satisfies equations (81) and (82), then $\eta^2 = f_1(t)M + f_3(x,t)$ and $\eta^3 = Ng_2(x,t) + g_3(x,t,E)$.*

Proof: Setting $D_1 \neq D_2$, from equation (82) we have $g(t) = 0$, which implies $\eta^3 = Ng_2(x,t) + g_3(x,t,E)$. From equations (80) and (83) we have $f_4(t) = 0$. Replacing it into equation (84), we have that $f_2(t) = 0$. So, $\eta^2 = f_1(t)M + f_3(x,t)$. ∎

Proposition 4.19. *If $\rho \neq 0$, η^1 satisfies equations (79), (88), (90), (92) and (93), η^3 satisfies Proposition 4.18 and equations (88), (90) and (92), then $\eta^1 = h(t)$ and $\eta^3 = Ng_2(t) + g_3(t, E)$.*

Proof: Substituting $\rho \neq 0$ in equation (93), we have $\eta^1_E = 0$, which implies $\eta^1 = h(x, t)$. The derivative of equation (90) with respect to x is given by

$$\rho(\eta^3_x - N\eta^3_{Nx}) - D_1\eta^3_{Ex} = 0. \tag{95}$$

Making the difference between (88) and (95), we have $D_1\eta^3_{Ex} + N\eta^3_{Nx} = 0$. Analysing these variables involved in the last equation and Proposition 4.18, we can affirm that $\eta^3_{xE} = 0$ and $\eta^3_{Nx} = 0$. Replacing these into equations (88) and (92), we obtain $\eta^3_x = 0$ and $\eta^1_x = 0$, respectively.

Since (79) holds and Proposition 4.18 is satisfied, we can conclude $\eta^1 = h(t)$ and $\eta^3 = Ng_2(t) + g_3(t, E)$. ■

Proposition 4.20. *If $\rho \neq 0$, $\delta \neq 0$, η^1 satisfies equation (91), η^2 satisfies (91) and (94), η^3 satisfies (87) and (94), and Propositions 4.18 and 4.19 hold, then $\eta^2 = -c_1M$, $\eta^1 = 0$, $\eta^3 = c_4N$ and $c_1\lambda = 0$. In particular, if $\lambda \neq 0$, then $\eta^2 = 0$, $\eta^1 = 0$, $\eta^3 = c_4N$, $\xi^1 = c_3$, and $\xi^2 = c_2$; and if $\mu \neq 0$, then $\eta^3 = -2c_1N$.*

Proof: Using the derivative of equation (91) with respect to E and M and the Propositions 4.18 and 4.19, we have:

$$\delta(\eta^2 + c_1M) = 0, \tag{96}$$

$$\delta(\eta^1 + Ef_1(t) + c_1E) = 0. \tag{97}$$

Considering $\delta \neq 0$ in equations (96) and (97), we have $\eta^2 = -c_1M$ and $\eta^1 = -E(f_1(t) + c_1)$, which implies $f_1(t) = -c_1$, $f_3(x, t) = 0$, since Propositions 4.18 and 4.19 hold. Thus, $\eta^1 = 0$.

Also, deriving equation (87) with respect to M and taking $\delta \neq 0$, we obtain $\eta^3_E = 0$. Replacing this into equation (87), we have that $\eta^3_t = 0$. So, $\eta^3 = c_4N$.

Besides that, the equation (94) can be rewritten as

$$\mu\eta^3 + 2c_1\mu N - c_1\lambda M = 0, \tag{98}$$

and its derivative with respect to M gives $c_1\lambda = 0$.

In particular, if $\lambda \neq 0$, then $c_1 = 0$. So, $\eta^2 = 0$, $\eta^1 = 0$, $\eta^3 = c_4 N$, $\xi^1 = c_3$, and $\xi^2 = c_2$. On the other hand, deriving equation (98) with respect to N, we have:

$$\mu(c_4 + 2c_1) = 0 \tag{99}$$

In particular, if $\mu \neq 0$, from equation (99) we can affirm $\eta^3 = -2c_1 N$. ∎

Therefore, the rewritten system for $p = 0, D_1 \neq D_2, \rho \neq 0, \delta \neq 0, \mu \neq 0$ and $\lambda \neq 0$ is:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0,$$

$$\eta^2 = 0,$$

$$\eta^3 = 0.$$

If $p = 0, D_1 \neq D_2, \rho \neq 0, \delta \neq 0, \mu \neq 0$ and $\lambda = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = 0,$$

$$\eta^2 = -c_1 M,$$

$$\eta^3 = -2c_1 N.$$

For $p = 0, D_1 \neq D_2, \rho \neq 0, \delta \neq 0, \mu = 0$ and $\lambda \neq 0$, we obtain the system :

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0,$$

$$\eta^2 = 0,$$

$$\eta^3 = c_4 N.$$

Furthermore, for $p = 0, D_1 \neq D_2, \rho \neq 0, \delta \neq 0, \mu = 0$ and $\lambda = 0$, the system becomes:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = 0,$$

$$\eta^2 = -c_1 M,$$

$$\eta^3 = c_4 N.$$

Proposition 4.21. *If $\rho \neq 0$ and $\delta = 0$, η^1 satisfies equation (91), η^2 satisfies equation (94), η^3 satisfies equation (87), and Propositions 4.18 and 4.19 hold, then $\eta^1 = c_4$, $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$, $\eta^3 = c_5 N + k e^{\rho E/D_1}$, where c_5, c_6, k are constants. In particular, if $\mu \neq 0$, then $\eta^2 = (c_5 + c_1)M + f_3(x,t)$ and $\eta^3 = c_5 N$.*

Proof: Assuming that Proposition 4.19 holds and $\delta = 0$ in equations (87) and (91), we obtain $\eta^3_t = 0$ and $\eta^1_t = 0$. Then $\eta^1 = c_4$ and $\eta^3 = c_5 N + g_4(E)$.

On the other hand, the equation (90) can be rewritten as

$$\rho g_4(E) - D_1 g_4'(E) = 0.$$

So, $g_4(E) = k e^{\rho E/D_1}$. Then $\eta^3 = c_5 N + k e^{\rho E/D_1}$.

Deriving equation (94) with respect to M and N and considering Proposition 4.18, we have, respectively:

$$f_1'(t) + c_1 \lambda = 0 \tag{100}$$

and

$$\mu(f_1(t) - c_5 - c_1) = 0, \tag{101}$$

which implies $f_1(t) = c_6 - c_1\lambda t$ and then $\mu(c_6 - c_5 - c_1 - c_1\lambda t) = 0$. From Proposition 4.18 and equation (100), we have $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$.

Therefore, equation (101) indicates that we can analyze this case for $\mu \neq 0$ and $\mu = 0$.

In particular, if $\mu \neq 0$, from equation (101) we can affirm that $c_6 = c_5 + c_1$ and $\lambda c_1 = 0$. Rewriting equation (94) we obtain $k = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$. So, $\eta^2 = (c_5 + c_1)M + f_3(x,t)$ and $\eta^3 = c_5 N$.

If $\mu = 0$, from equation (94) we also have $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t))$. ■

Thus, if $p = 0, D_1 \neq D_2, \rho \neq 0, \delta = 0, \mu \neq 0$ and $\lambda = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = (c_5 + c_1)M + f_3(x,t),$$

$$\eta^3 = c_5 N,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Setting the parameters as $p = 0, D_1 \neq D_2, \rho \neq 0, \delta = 0, \mu \neq 0$ and $\lambda \neq 0$, the system becomes:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = c_5 M + f_3(x,t),$$

$$\eta^3 = c_5 N,$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

On the other hand, $p = 0, D_1 \neq D_2, \rho \neq 0, \delta = 0$ and $\mu = 0$ lead us to the following system :

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = c_4,$$

$$\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t),$$

$$\eta^3 = c_5 N + ke^{\rho E/D_1},$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) = D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)).$$

Proposition 4.22. *If $\rho = 0$, η^1 and η^3 satisfy equations (87), (90) and (92), η^2 and η^3 satisfy equation (94), and Proposition 4.18 holds, then $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$ and $\eta^3 = c_7 N + g_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0$, $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$ and c_7 is a constant. In particular, if $\mu \neq 0$, then $\eta^2 = c_6 M + f_3(x,t)$ and $\eta^3 = (c_6 - c_1)N + g_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0$, $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$ and $\lambda c_1 = 0$; if $\mu = 0$, then $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$ and $\eta^3 = c_7 N + g_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$.*

Proof: Considering Proposition 4.18 and $\rho = 0$ in equations (90) and (92), we have $\eta^3_E = 0$ and $\eta^3_{xN} = 0$, which implies $\eta^3 = Ng_2(t) + g_3(x,t)$.

From equation (87), we have $Ng_2'(t) + \frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$. Thus, $g_2'(t) = 0$ and $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$, which implies $g_2(t) = c_7$ and $\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t))$, where c_7 is a constant.

Deriving equation (94) with respect to M we have:

$$f_1'(t) + c_1\lambda = 0,$$

which implies $f_1(t) = c_6 - c_1\lambda t$ then $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$ and we have the result.

Rewriting equation (94) we obtain

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) + \mu N(c_6 - c_7 - c_1\lambda t - c_1) = 0. \quad (102)$$

From equation (102), we can conclude that $\mu = 0$ or $c_6 - c_7 - c_1\lambda t - c_1 = 0$. So, $\mu = 0$ or $\mu \neq 0 \Rightarrow c_1\lambda = 0, c_7 = c_6 - c_1$. Then, we have the expected result. ∎

Proposition 4.23. *Consider* $\rho = 0$. *If* $\delta \neq 0$, η^1 *satisfies equations* (79) *and* (91), *and Proposition* 4.22 *is satisfied, then* $\eta^1_t = -E\delta f_3(x,t)$. *If* $\delta = 0$, η^1 *satisfies equations* (79) *and* (91), *then* $\eta^1 = h(x,E)$.

Proof: Consider $\delta \neq 0$. The following equations are the derivatives of (91) with respect to M and E, respectively:

$$\delta(\eta^1 + Ef_1(t) - E\eta^1_E + c_1 E) = 0, \tag{103}$$

$$\eta^1_{tE} + \delta[f_3(x,t) + M(f_1(t) - E\eta^1_{EE} + c_1)] = 0. \tag{104}$$

Since $\delta \neq 0$ and $f_1(t) = c_6 - c_1\lambda t$, equation (103) implies $\eta^1 + E(c_6 - c_1\lambda t - \eta^1_E + c_1) = 0$. Deriving this last equation with respect to t and E, we have, respectively:

$$\eta^1_t + E(-c_1\lambda - \eta^1_{Et}) = 0, \tag{105}$$

$$E\eta^1_{EE} = c_6 - c_1\lambda t + c_1. \tag{106}$$

Substituting equation (106) into equation (104), we have

$$\eta^1_{tE} + \delta f_3(x,t) = 0. \tag{107}$$

Also, the derivative of equations (105) and (107) with respect to E give $c_1\lambda + E\eta^1_{tEE} = 0$ and $\eta^1_{tEE} = 0$. So, $c_1\lambda = 0$, and replacing it into equation (103) and resolving this new equation it by integrating factor, we can affirm

$$\eta^1 = ((c_6 + c_1)\ln E + f_5(x,t))E. \tag{108}$$

Taking into account equations (105), (107) and (108), then $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$.
If $\delta = 0$, $\eta^1 = h(x,E)$ comes directly from equations (79) and (91). ■

Hence, $p = 0, D_1 \neq D_2, \rho = 0, \mu \neq 0, \delta \neq 0$ and $\lambda = 0$ give the system :

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = E((c_6 + c_1)\ln E + f_5(x,t)),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = (c_6 - c_1)N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1 \frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

For $p = 0, D_1 \neq D_2, \rho = 0, \mu \neq 0, \delta \neq 0$ and $\lambda \neq 0$, we can rewrite the system as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = E(c_6 \ln E + f_5(x,t)),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_6 N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2 \frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1 \frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

Since $p = 0, D_1 \neq D_2, \rho = 0, \mu \neq 0, \delta = 0$ and $\lambda = 0$, the system rewritten is:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = (c_6 - c_1)N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

Let $p = 0, D_1 \neq D_2, \rho = 0, \mu \neq 0, \delta = 0$ and $\lambda \neq 0$, then the system becomes :

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_6 N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(x,t) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

Considering $p = 0, D_1 \neq D_2, \rho = 0, \mu = 0,$ and $\delta \neq 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = E((c_6 + c_1)\ln E + f_5(x,t)),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_7 N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

Setting $p = 0, D_1 \neq D_2, \rho = 0, \mu = 0$ and $\delta = 0$, we obtain the following system :

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t),$$

$$\eta^3 = c_7 N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) = D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)).$$

The case *$D_1 = D_2$:*

In addition to the condition $p = 0$, we add the condition $D_1 = D_2$.

Proposition 4.24. *If $\rho \neq 0$, η^2 satisfies (80), (84), (85), (93) and (94), η^3 satisfies equations (81), (89) and (94), and it is also as the result of the Proposition 4.19, then $\eta^1 = h(t)$, $\eta^2 = (c_6 - c_1\lambda t)M + f_3(x,t)$ and $\eta^3 = Ng_2(t) + g_3(t,E)$.*

Proof: Let $\rho \neq 0$. From equations (80) and (85) we have $f_4(t) = 0$. Replacing it into equation (84), we have $f_2(t) = 0$. So, $\eta^2 = f_1(t)M + f_3(x,t)$. From equations (81) and (89) we have $g(t) = 0$, which implies $\eta^3 = Ng_2(x,t) + g_3(x,t,E)$.

Those last two results are the same in the Proposition 4.18 without forcing conditions about D_1 and D_2. It is also worth to stress that Proposition 4.19 does not use as well the condition about D_1 and D_2, only $p = 0$, $\rho \neq 0$ and the result of Proposition 4.18.

So, we have the same conclusion of Proposition 4.19, i.e., $\eta^1 = h(t)$ and $\eta^3 = Ng_2(t) + g_3(t, E)$.

Deriving equation (94) with respect to M, we have $f_1'(t) + c_1\lambda = 0$, which implies $f_1(t) = c_6 - c_1\lambda t$.

Thus, we have the desired outcome. ■

Considering the conditions and results of Proposition 4.24, we can derive equation (94) with respect to N, so

$$\mu(c_6 - g_2(t) - c_1\lambda t - c_1) = 0. \tag{109}$$

This result suggests that we can divide this case into $\mu \neq 0$ and $\mu = 0$.

Using the result in equation (109) into equation (94), we found

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) - \mu g_3(t,E) = 0. \tag{110}$$

Proposition 4.25. *If $\rho \neq 0$, $\mu \neq 0$, Proposition 4.24 is satisfied, η^1 satisfies equations (87), (90) and (91), η^2 satisfies equation (91) and η^3 satisfies equations (87) and (90), then $\eta^2 = c_6M + f_3(x,t)$ and $\eta^3 = N(c_6 - c_1)$. In particular, if $\lambda \neq 0$, then $c_1 = 0$; if $\lambda = 0$ then $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$; if $\delta \neq 0$, then $\eta^1 = 0$, $\eta^2 = -c_1M$ and $\eta^3 = -2c_1N$; if $\delta = 0$, then $\eta^1 = c_4$.*

Proof: If $\mu \neq 0$ into (110), then $g_{3_E} = 0$, i.e.,

$$g_3 = g_3(t) \tag{111}$$

and

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0. \tag{112}$$

Also if Proposition 4.24 holds and $\mu \neq 0$, from equation (109) we obtain $c_6 - c_1 - g_2(t) - c_1\lambda t = 0$, and replacing $\eta^1_E = 0$ and equation (111) into (90) we have $\eta^3 = N\eta^3_N$, i.e., $g_3(t) = 0$.

Using Proposition 4.24 and equation (87) we have $\eta^3_t = 0$, which implies $c_1\lambda = 0$ by equation (109), and that means $g_2 = c_6 - c_1$. In particular, if $\lambda \neq 0$, then $c_1 = 0$; and if $\lambda = 0$, equation (112) gives $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$. So far, we have $\eta^2 = c_6M + f_3(x,t)$ and $\eta^3 = N(c_6 - c_1)$.

On the other hand, we can derive equation (91) with respect to M and E, respectively:

$$\delta(h(t) + c_6E + c_1E) = 0, \tag{113}$$

$$\delta(\eta^2 + c_1M) = 0. \tag{114}$$

In particular, if we consider $\delta \neq 0$ into equations (113) and (114), we can conclude that $h(t) = 0$, $c_6 = -c_1$ and $\eta^2 = c_6M$.

If $\delta = 0$, from equation (91) we have $\eta^1_t = 0$, then $\eta^1 = c_4$.

Therefore, we have the result. ■

Thus, if $p = 0, D_1 = D_2, \rho \neq 0, \mu \neq 0, \lambda \neq 0$ and $\delta = 0$, we can rewrite the system as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = c_4, \quad \eta^2 = c_6M + f_3(x,t),$$
$$\eta^3 = c_6N,$$

where
$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Since $p = 0, D_1 = D_2, \rho \neq 0, \mu \neq 0, \lambda = 0$ and $\delta = 0$, the system remains as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1t + c_2,$$

$$\eta^1 = c_4, \quad \eta^2 = c_6M + f_3(x,t),$$

$$\eta^3 = (c_6 - c_1)N,$$

where
$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

For $p = 0, D_1 = D_2, \rho \neq 0, \mu \neq 0, \lambda \neq 0$ and $\delta \neq 0$, the rewritten system is:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = 0.$$

On the other hand, $p = 0, D_1 = D_2, \rho \neq 0, \mu \neq 0, \lambda = 0$ and $\delta \neq 0$ give the system

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = 0, \quad \eta^2 = -c_1 M, \quad \eta^3 = -2c_1 N.$$

Proposition 4.26. *If $\rho \neq 0$, $\mu = 0$, $\delta \neq 0$, Proposition 4.24 is satisfied, $\eta^1 = 0$ and η^2 satisfy equation (91), then $\eta^1 = 0$, $\eta^2 = -c_1 M$ and $\eta^3 = c_7 N$. In particular, if $\lambda \neq 0$, then $c_1 = 0$.*

Proof: Consider $\delta \neq 0$ and deriving the equation (91) with respect to M and E, we have, respectively:

$$\eta^1 = E(c_1 \lambda t - c_1 - c_6), \tag{115}$$

$$\eta^2 = -c_1 M.$$

Since $\eta^1 = h(t)$, equation (115) implies $\eta^1 = 0$, $c_6 = -c_1$ and $c_1 \lambda = 0$; in particular, if $\lambda \neq 0$, then $c_1 = 0$. Therefore, the expected result has been achieved. ∎

Proposition 4.27. *If $\rho \neq 0$, $\mu = 0$, $\delta = 0$, Proposition 4.24 is satisfied and η^1 satisfies equations (87), (90) and (91), η^2 satisfies equation (91) and η^3 satisfies equations (87) and (90), then $\eta^1 = c_4$ and $\eta^3 = c_7 N + k e^{\rho E / D_1}$.*

Proof: Consider that Proposition 4.24 holds and $\delta = 0$ into equations (87) and (91). So, we have $\eta^3_t = 0$ and $\eta^1_t = 0$, respectively, which implies $\eta^3 = c_7 N + g_3(E)$ and $\eta^1 = c_4$.

Besides that, equation (90) gives $\rho g_3(E) - D_1 g_3'(E) = 0$, then $g_3(E) = k e^{\rho E / D_1}$.

Thus, the proof has been done. ∎

Hence, $p = 0, D_1 = D_2, \rho \neq 0, \mu = 0, \delta \neq 0$ and $\lambda = 0$ lead to the system

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = 0, \quad \eta^2 = -c_1 M, \quad \eta^3 = c_7 N.$$

If $p = 0, D_1 = D_2, \rho \neq 0, \mu = 0, \delta \neq 0$ and $\lambda \neq 0$ the system becomes :

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = c_7 N.$$

Considering $p = 0, D_1 = D_2, \rho \neq 0, \mu = 0, \delta = 0$ and $\lambda = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = c_4, \quad \eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_7 N + ke^{\rho E/D_1},$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

And making $p = 0, D_1 = D_2, \rho \neq 0, \mu = 0, \delta = 0$ and $\lambda \neq 0$, we obtain the system

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = c_4, \quad \eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_7 N + ke^{\rho E/D_1},$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Proposition 4.28. *If $\rho = 0$, η^2 satisfies (80), (84) and (94), η^3 satisfies (81), (87), (90), (92) and (94), then $\eta^2 = f_1(t)M + f_3(x,t) + f_4(t)N$ and $\eta^3 = (ke^{\lambda t})M + Ng_2(t) + g_3(x,t)$. In particular, if $\lambda = 0$, then $\eta^3 = kM + Ng_2(t) + g_3(x,t)$.*

Proof: Substituting $\rho = 0$ in equations (84), (90) and (92) we have $f_2(t) = 0$, $\eta^3_E = 0$ and $\eta^3_{xN} = 0$, which implies $\eta^2 = f_1(t)M + f_3(x,t) + f_4(t)N$ and $\eta^3 = g(t)M + Ng_2(t) + g_3(x,t)$ by equations (80) and (81).

Replacing $\rho = 0$ and $\eta^3_E = 0$ into equation (87) and calculating its derivative with respect to M, we have $g'(t) = \lambda g(t)$. Hence, $g(t) = ke^{\lambda t}$. Particularly, if $\lambda = 0$, then $g(t) = k$. The result suggests that we can split this case into $\lambda \neq 0$ and $\lambda = 0$. ■

Before analyzing other equations of the system (78)-(94) for $\lambda \neq 0$ and $\lambda = 0$, and assuming the Proposition 4.28 holds, we will present some of these equations rewritten or its derivatives, which can assist the analysis for both cases of λ.

So, we can derive equation (87) with respect to N:

$$g_2'(t) = -\mu g(t). \tag{116}$$

Conversely, deriving equation (94) with respect to M and N, respectively:

$$f_1'(t) = \mu g(t) - c_1\lambda, \tag{117}$$

$$f_4'(t) + \lambda f_4(t) - \mu g_2(t) + \mu f_1(t) - \mu c_1 = 0. \tag{118}$$

Proposition 4.29. *If $\rho = 0$, $\lambda \neq 0$, η^1 satisfies equations (79), (87) and (91), η^2 satisfies equation (91), and the Proposition 4.28 is satisfied, then $\eta^1 = h(x,t,E)$, $\eta^2 = f_3(x,t) + \left(\frac{\mu}{\lambda}\left(c_5 + c_1 - c_6 - \frac{\mu k}{\lambda}e^{\lambda t} + c_1\lambda t\right) - \mu c_1 + \frac{k_1}{e^{\lambda t}}\right)N + \left(\frac{\mu k}{\lambda}e^{\lambda t} + c_6 - c_1\lambda t\right)M$ and $\eta^3 = (ke^{\lambda t})M + N\left(c_5 - \frac{\mu k}{\lambda}e^{\lambda t}\right) + g_3(x,t)$, where $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$. Particularly, if $\mu = 0$, then $\eta^2 = (c_6 - c_1\lambda t)M + \frac{k_1}{e^{\lambda t}}N + f_3(x,t)$ and $\eta^3 = (ke^{\lambda t})M + c_5N + g_3(x,t)$, where $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$; if $\delta = 0$, then $\eta^1 = h(x,E)$.*

Proof: In view of Proposition 4.28 and setting $\lambda \neq 0$ in equations (116) and (117), we obtain, respectively:

$$g_2(t) = c_5 - \frac{\mu k}{\lambda}e^{\lambda t}, \tag{119}$$

$$f_1(t) = \frac{\mu k}{\lambda}e^{\lambda t} - c_1\lambda t + c_6. \tag{120}$$

Accordingly, the condition

$$f_4'(t) + \lambda f_4(t) + \mu\left(\frac{\mu k}{\lambda}e^{\lambda t} + c_6 + \frac{\mu k}{\lambda}e^{\lambda t} - c_1\lambda t - c_5 - c_1\right) = 0 \tag{121}$$

comes directly from equation (118), replacing (119) and (120) into it. Thus,

$$f_4(t) = \frac{\mu}{\lambda}\left(c_5 - \frac{\mu k}{\lambda}e^{\lambda t} - c_6 + c_1 + c_1\lambda t\right) + \frac{k_1}{e^{\lambda t}} - \mu c_1. \quad (122)$$

Besides that, rewriting the equations (87) and (94), we have the conditions

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$$

and

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0,$$

respectively.

Particularly, if $\mu = 0$, then equation (122) gives $f_4(t) = \frac{k_1}{e^{\lambda t}}$, which implies $\eta^2 = (c_6 - c_1\lambda t)M + \frac{k_1}{e^{\lambda t}}N + f_3(x,t)$ and $\eta^3 = (ke^{\lambda t})M + Nc_5 + g_3(x,t)$, $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$.

In particular, if $\delta = 0$, then equations (79) and (91) give $\eta^1 = h(x,E)$. ■

Proposition 4.30. *If $\rho = 0$, $\lambda \neq 0$, $\delta \neq 0$, ξ^1 and ξ^2 satisfy equations (78), η^1 satisfies equations (79) and (91), η^2 satisfies equation (91), and the Proposition (4.29) is satisfied, then $\xi^1 = c_3$, $\xi^2 = c_2$, $\eta^1 = ((c_6 + c_1)\ln E + f_5(x,t))E$, $\eta^2 = c_6 M + f_3(x,t)$ and $\eta^3 = (ke^{\lambda t})M + Nc_5 + g_3(x,t)$, where $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$ and $\mu(-c_5 + c_6) = 0$. Particularly, if $\mu \neq 0$, then $\eta^3 = Nc_6 + g_3(x,t)$; if $\mu = 0$, then the additional conditions on η^1, η^2 and η^3 become $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$.*

Proof: In view of Proposition 4.29, equation (91) becomes

$$\eta^1_t + \delta E f_3(x,t) = -\delta M\left(\eta^1 + E\frac{\mu k}{\lambda}e^{\lambda t} - Ec_1\lambda t + Ec_6 - E\eta^1_E + c_1 E\right) - \delta E N f_4(t). \quad (123)$$

Considering equation (79), $\delta \neq 0$ into equation (123) and knowing that the variables of f_3 are x and t, we obtain

$$f_4(t) = 0, \quad (124)$$

$$\eta^1_t = -\delta E f_3(x,t), \quad (125)$$

$$\eta^1 = E\left(c_1\lambda t + \eta^1_E - \frac{\mu k}{\lambda}e^{\lambda t} - c_6 - c_1\right). \quad (126)$$

Comparing equation (125) to the derivatives of equations (125) with respect to E and (126) with respect to t, we can affirm $\mu k e^{\lambda t} = c_1\lambda$, since $\lambda \neq 0$, that implies $\mu k = 0$ and $c_1 = 0$. Analyzing equations (78), this last condition implies $\xi^1 = c_3$ and $\xi^2 = c_2$.

Accordingly, resolving equation (126) by integrating factor, we can affirm

$$\eta^1 = ((c_6 + c_1)\ln E + f_5(x,t))E. \tag{127}$$

From equations (125), (126) and (127) we obtain $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$.

On the other hand, the condition $\mu(c_6 - c_5) = 0$ comes directly from replacing $\mu k = 0$, $c_1 = 0$ and (124) into equation (121).

So, if $\mu \neq 0$, then $k = 0$ and $c_5 = c_6$. Consequently, $\eta^3 = Nc_6 + g_3(x,t)$.

Particularly, if $\mu = 0$, then the additional conditions on η^1, η^2 and η^3 become $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$. ■

Proposition 4.31. *If $\rho = 0$, $\lambda = 0$, η^1 satisfies equations (79), (87) and (91), η^2 satisfies equation (91), and the Proposition (4.28) is satisfied, then $\eta^1 = h(x,t,E)$, $\eta^2 = (\mu kt + c_{11})M + f_3(x,t) + f_4(t)N$ and $\eta^3 = kM + N(c_{10} - \mu kt) + g_3(x,t)$, where $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$, $\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$ and c_{10}, c_{11} are arbitrary constants. In particular, if $\delta = 0$, then $\eta^1 = h(x,E)$; if $\mu = 0$, then $\eta^2 = c_{11}M + Nk_1 + f_3(x,t)$ and $\eta^3 = kM + Nc_{10} + g_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$, $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and k_1 is a constant.*

Proof: In view of Proposition 4.28 and setting $\lambda = 0$ in equations (116) and (117), we obtain, respectively:

$$g_2(t) = c_{10} - \mu kt, \tag{128}$$

$$f_1(t) = \mu kt + c_{11}, \tag{129}$$

where c_{10}, c_{11} are constants.

Accordingly, the condition

$$f_4'(t) + \mu(2\mu kt - c_{10} + c_{11} - c_1) = 0 \tag{130}$$

comes directly from equation (118), replacing (128) and (129) into it. Then,

$$f_4(t) = \mu(c_{10} - c_{11} + c_1)t - \mu^2 kt^2 + k_1,$$

where k_1 is a constant.

Besides that, rewriting equations (87) and (94), we have the conditions

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0 \tag{131}$$

and

$$\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0, \tag{132}$$

respectively.

In particular, if $\delta = 0$, then equations (79) and (91) give $\eta^1 = h(x, E)$.

If $\mu = 0$, then $\eta^2 = c_{11}M + k_1N + f_3(x,t)$ and $\eta^3 = kM + c_{10}N + g_3(x,t)$, where $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$. ■

Proposition 4.32. *If $\rho = 0$, $\lambda = 0$, $\delta \neq 0$, η^1 satisfies equations (79) and (91), and the Proposition (4.31) is satisfied, then $\eta^1 = ((c_{11} + c_1)\ln E + f_5(x,t))E$, $\eta^2 = c_{11}M + f_3(x,t)$ and $\eta^3 = kM + c_{10}N + g_3(x,t)$, where $\mu(c_{11} - c_{10} - c_1) = 0$, $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$, $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$. Particularly, if $\mu \neq 0$, then $\eta^3 = c_{10}N + g_3(x,t)$; if $\mu = 0$, then the only additional conditions on η^1, η^2 and η^3 are $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$.*

Proof: In view of Proposition 4.31, equation (91) becomes

$$\eta^1_t + \delta E f_3(x,t) = -\delta M(\eta^1 + E\mu kt + Ec_{11} - E\eta^1_E + c_1E) - \delta EN f_4(t). \tag{133}$$

Considering equation (79), $\delta \neq 0$ into equation (133) and knowing that the variables of f_3 are x and t, we obtain

$$f_4(t) = 0, \tag{134}$$

$$\eta^1_t = -\delta E f_3(x,t), \tag{135}$$

$$\eta^1 = E(\eta^1_E - \mu kt - c_{11} - c_1). \tag{136}$$

Accordingly, the conditions $\mu(c_{11} - c_{10} - c_1) = 0$ and $\mu k = 0$ come directly from equation (130) whereas (134) holds. So, if $\mu \neq 0$, then $k = 0$ and, consequently, $\eta^3 = c_{10}N + g_3(x,t)$.

On the other hand, resolving equation (136) by integrating factor, we can affirm

$$\eta^1 = ((c_{11} + c_1)\ln E + f_5(x,t))E. \tag{137}$$

Equations (135), (136) and (137) give $\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t))$.

Since equations (131) and (132) still hold, so $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$.

Particularly, if $\mu = 0$, then the additional conditions on η^1, η^2 and η^3 become $\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0$ and $\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0$. ■

Thus, if $p = 0, D_1 = D_2, \rho = 0, \lambda \neq 0, \delta \neq 0$ and $\mu \neq 0$, we can rewrite the system as:

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = E(c_6 \ln E + f_5(x,t)),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = c_6 N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_1\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Setting $p = 0, D_1 = D_2, \rho = 0, \lambda \neq 0, \delta \neq 0$ and $\mu = 0$, we obtain the system

$$\xi^1 = c_3, \quad \xi^2 = c_2,$$

$$\eta^1 = E(c_6 \ln E + f_5(x,t)),$$

$$\eta^2 = c_6 M + f_3(x,t),$$

$$\eta^3 = (ke^{\lambda t})M + c_5 N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

For $p = 0, D_1 = D_2, \rho = 0, \lambda \neq 0, \delta = 0$ and $\mu \neq 0$ the system becomes

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = \left(\frac{\mu k}{\lambda}e^{\lambda t} - c_1\lambda t + c_6\right)M + \left(\frac{\mu}{\lambda}\left(c_5 - \frac{\mu k}{\lambda}e^{\lambda t} + c_1 + c_1\lambda t - c_6\right) + \frac{k_1}{e^{\lambda t}} - \mu c_1\right)N + f_3(x,t),$$

$$\eta^3 = (ke^{\lambda t})M + \left(c_5 - \frac{\mu k}{\lambda}e^{\lambda t}\right)N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Taking $p = 0, D_1 = D_2, \rho = 0, \lambda \neq 0, \delta = 0$ and $\mu = 0$, the rewritten system is:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = (c_6 - c_1\lambda t)M + k_1 e^{-\lambda t}N + f_3(x,t),$$

$$\eta^3 = (ke^{\lambda t})M + c_5 N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) + \lambda f_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

If $p = 0, D_1 = D_2, \rho = 0, \lambda = 0, \delta \neq 0$ and $\mu \neq 0$, then the system remains as following:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = E((c_{11} + c_1)\ln E + f_5(x,t)),$$

$$\eta^2 = c_{11}M + f_3(x,t),$$

$$\eta^3 = (c_{11} - c_1)N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Considering $p = 0, D_1 = D_2, \rho = 0, \lambda = 0, \delta \neq 0$ and $\mu = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = E((c_{11} + c_1)\ln E + f_5(x,t)),$$

$$\eta^2 = c_{11}M + f_3(x,t),$$

$$\eta^3 = kM + c_{10}N + g_3(x,t),$$

where

$$\delta f_3(x,t) = -\frac{\partial}{\partial t}(f_5(x,t)),$$

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

On the other hand, $p = 0, D_1 = D_2, \rho = 0, \lambda = 0, \delta = 0$ and $\mu \neq 0$ imply the system rewritten as following:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = (\mu k t + c_{11})M + (k_1 - \mu^2 k t^2 + \mu(c_{10} - c_{11} + c_1)t)N + f_3(x,t),$$

$$\eta^3 = kM + (c_{10} - \mu k t)N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(f_3(x,t)) - \mu g_3(x,t) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0.$$

Setting the parameters as $p = 0, D_1 = D_2, \rho = 0, \lambda = 0, \delta = 0$ and $\mu = 0$, we can rewrite the system as:

$$\xi^1 = \frac{c_1}{2}x + c_3, \quad \xi^2 = c_1 t + c_2,$$

$$\eta^1 = h(x,E),$$

$$\eta^2 = c_{11}M + k_1 N + f_3(x,t),$$

$$\eta^3 = kM + c_{10}N + g_3(x,t),$$

where

$$\frac{\partial}{\partial t}(f_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(f_3(x,t)) = 0,$$

$$\frac{\partial}{\partial t}(g_3(x,t)) - D_2\frac{\partial^2}{\partial x^2}(g_3(x,t)) = 0.$$

5 INVARIANT SOLUTIONS: A CONSTRUCTION

In this chapter we use the Lie point symmetries and the Invariant Form Method presented in chapter 3 to construct explicit invariant solutions to the system.

Based on Invariant Form Method we obtain an ordinary differential equations (ODE) system but unfortunately not all the obtained ODE systems are easy to be solved, so we will show in the following chapter the results considering some of the generators associated with the system (5).

Hence, several graphics were made using parameters from [1], [3], [9] and [25]. Some of them, such as μ, λ and δ, as mentioned in chapter 2, are estimations supported by the literature and not obtained from experimental data until now.

5.1 GENERAL CASE

The system (5) is given by

$$\begin{cases} N_t &= D_1(N^p N_{xx} + pN^{p-1}N_x^2) - \rho(N_x E_x + N E_{xx}), \\ E_t &= -\delta M E, \\ M_t &= D_2 M_{xx} + \mu N - \lambda M. \end{cases}$$

We notice from section 4.2 that the translations $X_1 = \partial_t$ and $X_2 = \partial_x$ are generators related to system (5) for all parameters. Thus, the linear combination of X_1 and X_2 as $X_1 + cX_2$, where c is an arbitrary constant, is a common generator in all cases as well.

Considering the invariants construction process and the Example 3.12, we have to solve the system (33), rewritten here:

$$\begin{cases} -c\Phi_1' &= D_1(\Phi_1^p \Phi_1'' + p\Phi_1^{p-1}\Phi_1'^2) - \rho(\Phi_1'\Phi_2' + \Phi_1\Phi_2''), \\ c\Phi_2' &= \delta\Phi_3\Phi_2, \\ -c\Phi_3' &= D_2\Phi_3'' + \mu\Phi_1 - \lambda\Phi_3. \end{cases}$$

In order to do so, we can divide this new system into some subcases as follows:

1. $p \neq 0$, $\lambda \neq 0$ and $\mu = 0$:

   Taking $\mu = 0$ and $\lambda \neq 0$ in third equation of (33), we obtain

$$\Phi_3 = c_1 e^{\frac{1}{2}w\left(-\frac{\sqrt{c^2+4D_2\lambda}}{D_2}-\frac{c}{D_2}\right)} + c_2 e^{\frac{1}{2}w\left(\frac{\sqrt{c^2+4D_2\lambda}}{D_2}-\frac{c}{D_2}\right)}. \tag{138}$$

   Then, using this result into second equation of (33), we have

$$\Phi_2 = c_3 e^{\frac{\delta D_2}{c}\left(\frac{2c_2 e^{\frac{w\left(\sqrt{c^2+4D_2\lambda}-c\right)}{2D_2}}}{\sqrt{c^2+4D_2\lambda}-c} - \frac{2c_1 e^{-\frac{w\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c+\sqrt{c^2+4D_2\lambda}}\right)}. \tag{139}$$

   At last, we can solve the first equation of (33) considering $p \neq 0$ and then

$$\Phi_1 = \left(\frac{p}{D_1}\right)^{\frac{1}{p}} \left(-cw + k + c_3\rho e^{\frac{\delta}{c}\left(-\frac{2c_1 D_2 e^{-\frac{w\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c+\sqrt{c^2+4D_2\lambda}} - \frac{2c_2 D_2 e^{-\frac{w\left(c-\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c-\sqrt{c^2+4D_2\lambda}}\right)}\right)^{1/p}.$$

   Now taking $\alpha_1 = \frac{c+\sqrt{c^2+4D_2\lambda}}{2D_2}$ and $\alpha_2 = \frac{c-\sqrt{c^2+4D_2\lambda}}{2D_2}$, knowing that $w = x - ct$ and (32) holds, we find the solution to system (5) when $p \neq 0$, $\lambda \neq 0$ and $\mu = 0$:

$$\begin{cases} N(x,t) &= \left(\frac{p}{D_1}\right)^{\frac{1}{p}} \left(-cx + c^2 t + k + c_3\rho e^{-\frac{\delta}{c}\left(\frac{c_1 e^{-\alpha_1(x-ct)}}{\alpha_1} + \frac{c_2 e^{-\alpha_2(x-ct)}}{\alpha_2}\right)}\right)^{1/p}, \\ E(x,t) &= c_3 e^{-\frac{\delta}{c}\left(\frac{c_2 e^{-\alpha_2(x-ct)}}{\alpha_2} + \frac{c_1 e^{-\alpha_1(x-ct)}}{\alpha_1}\right)}, \\ M(x,t) &= c_1 e^{-\alpha_1(x-ct)} + c_2 e^{-\alpha_2(x-ct)}. \end{cases} \tag{140}$$

2. $p = 0$, $\lambda \neq 0$ and $\mu = 0$:

   Let $\mu = 0$ and $\lambda \neq 0$. Then Φ_3 and Φ_2 are given by (138) and (139), respectively.

Now, considering $p = 0$ into the first equation of (33), we obtain

$$\Phi_1 = e^{\left(k - \frac{cw}{D_1} + \frac{c_3 \rho}{D_1} e^{\left(\frac{\delta D_2}{c}\left(\frac{2c_2 e^{\frac{w\left(\sqrt{c^2+4D_2\lambda}-c\right)}{2D_2}}}{\sqrt{c^2+4D_2\lambda}-c} - \frac{2c_1 e^{-\frac{w\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c+\sqrt{c^2+4D_2\lambda}}\right)\right)}\right)}.$$

Knowing that $w = x - ct$ and (32) holds, we find the solution to system (5) when $\mu = 0, \lambda \neq 0$ and $p = 0$:

$$\begin{aligned}
N(x,t) &= e^{\left(k - \frac{c(x-ct)}{D_1} + \frac{c_3 \rho}{D_1} e^{\left(\frac{\delta D_2}{c}\left(\frac{2c_2 e^{\frac{(x-ct)\left(\sqrt{c^2+4D_2\lambda}-c\right)}{2D_2}}}{\sqrt{c^2+4D_2\lambda}-c} - \frac{2c_1 e^{-\frac{(x-ct)\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c+\sqrt{c^2+4D_2\lambda}}\right)\right)}\right)}, \\
E(x,t) &= c_3 e^{\frac{\delta D_2}{c}\left(\frac{2c_2 e^{\frac{(x-ct)\left(\sqrt{c^2+4D_2\lambda}-c\right)}{2D_2}}}{\sqrt{c^2+4D_2\lambda}-c} - \frac{2c_1 e^{-\frac{(x-ct)\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}}{c+\sqrt{c^2+4D_2\lambda}}\right)}, \\
M(x,t) &= c_1 e^{\frac{1}{2}(x-ct)\left(-\frac{\sqrt{c^2+4D_2\lambda}}{D_2} - \frac{c}{D_2}\right)} + c_2 e^{\frac{1}{2}(x-ct)\left(\frac{\sqrt{c^2+4D_2\lambda}}{D_2} - \frac{c}{D_2}\right)}.
\end{aligned}$$

More directly, we present the solutions for the other cases below:

3. $p = 0$, $\lambda = 0$ and $\mu = 0$:

$$N(x,t) = e^{\frac{c_3 \rho e^{\frac{\delta\left(\frac{c_1 D_2^2 e^{-\frac{c(x-ct)}{D_2}}}{c} + c c_2 (x-ct)\right)}{c^2}}}{D_1} - \frac{c(x-ct)}{D_1} + k}$$

$$E(x,t) = c_3 e^{\frac{\delta\left(c^2 c_2 (x-ct) + c_1 D_2^2 e^{-\frac{c(x-ct)}{D_2}}\right)}{c^3}}$$

$$M(x,t) = c_2 - \frac{c_1 D_2 e^{-\frac{c(x-ct)}{D_2}}}{c}$$

4. $p = 0$, $\lambda \neq 0$ and $\rho = 0$:

$$N(x,t) = c_2 - \frac{c_1 D_1 e^{-\frac{c(x-ct)}{D_1}}}{c}$$

$$E(x,t) = c_5 e^{\frac{\delta(c_2\mu(x-ct))}{c\lambda} + \frac{\delta\left((c+\alpha)c_4 e^{\frac{(x-ct)(\alpha-c)}{2D_2}} + (c-\alpha)c_3 e^{-\frac{(x-ct)(c+\alpha)}{2D_2}}\right)}{2c\lambda} + \frac{\delta c_1 D_1^4 \mu e^{-\frac{c(x-ct)}{D_1}}}{c^3(c^2(D_1-D_2)+D_1^2\lambda)}}$$

$$M(x,t) = \frac{-c_1 D_1^3 \mu e^{-\frac{c(x-ct)}{D_1}}}{c(c^2(D_1-D_2)+D_1^2\lambda)} + c\left(c_2\mu e^{\frac{(x-ct)(c+\alpha)}{2D_2}} + c_4\lambda e^{\frac{(x-ct)\alpha}{D_2}} + c_3\lambda\right) e^{-\frac{(x-ct)(c+\alpha)}{2D_2}},$$

where $\alpha = \sqrt{c^2 + 4D_2\lambda}$

5. $p = 0$, $\lambda = 0$, $\rho = 0$ and $D_1 \neq D_2$:

$$N(x,t) = c_2 - \frac{c_1 D_1 e^{-\frac{c(x-ct)}{D_1}}}{c}$$

$$E(x,t) = c_4 e^{\frac{\delta\left(-c^2 c_3 D_2 e^{-\frac{c(x-ct)}{D_2}} - c^2 k(x-ct) - \frac{1}{2}c^2 c_2 \mu (x-ct)^2 + \frac{c_1 D_1^4 \mu e^{-\frac{c(x-ct)}{D_1}}}{c(D_1-D_2)} + c c_2 D_2 \mu (x-ct)\right)}{c^4}}$$

$$M(x,t) = e^{-\frac{c(x-ct)}{D_2}}\left(c_3 - \frac{e^{c(x-ct)\left(\frac{1}{D_2}-\frac{1}{D_1}\right)}\left(\frac{c_1 D_1^3 D_2 \mu}{cD_1 - cD_2} + D_2 e^{\frac{c(x-ct)}{D_1}}(c(k+c_2\mu(x-ct)) - c_2 D_2 \mu)\right)}{c^2 D_2}\right)$$

6. $p = 0$, $\lambda = 0$, $\rho = 0$ and $D_1 = D_2$:

$$N(x,t) = c_2 - \frac{c_1 D_2 e^{-\frac{c(x-ct)}{D_2}}}{c}$$

$$E(x,t) = c_4 e^{\frac{\delta e^{-\frac{c(x-ct)}{D_2}}\left(-c^3\left((x-ct)e^{\frac{c(x-ct)}{D_2}}(2k+c_2\mu(x-ct)) + 2c_3 D_2\right) + 2c^2 c_2 D_2 \mu (x-ct) e^{\frac{c(x-ct)}{D_2}} + 2c c_1 D_2^2 \mu (x-ct) + 2 c_1 D_2^3 \mu\right)}{2c^5}}$$

$$M(x,t) = -\frac{c^2 c_3\left(-e^{-\frac{c(x-ct)}{D_2}}\right) + c_1 D_2 \mu (x-ct) e^{-\frac{c(x-ct)}{D_2}} + ck + c c_2 \mu (x-ct) - c_2 D_2 \mu}{c^2}$$

7. $p = 0$, $D_1 = D_2$, $\lambda \neq 0$ and $\delta = 0$:

$$N(x,t) = c_2 - \frac{c_1 D_2 e^{-\frac{c(x-ct)}{D_2}}}{c}$$

$$E(x,t) = c_1$$

$$M(x,t) = \frac{c\lambda e^{-\frac{(x-ct)\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}}\left(c_4 e^{\frac{(x-ct)\sqrt{c^2+4D_2\lambda}}{D_2}} + c_3\right) + c_1(-D_2)\mu e^{-\frac{c(x-ct)}{D_2}} + c c_2 \mu}{c\lambda}$$

8. $p = 0$, $\lambda \neq 0$ and $\delta = 0$:

$$N(x,t) = c_2 - \frac{c_1 D_1 e^{-\frac{c(x-ct)}{D_1}}}{c}$$

$$E(x,t) = c_1$$

$$M(x,t) = \left(c(c^2(D_1 - D_2) + D_1^2\lambda) \left(c_2 \mu e^{\frac{(x-ct)\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2}} + c_4 \lambda e^{\frac{(x-ct)\sqrt{c^2+4D_2\lambda}}{D_2}} + c_3\lambda \right) \right.$$
$$\left. -c_1 D_1^3 \lambda \mu e^{\frac{(x-ct)\left(c+\sqrt{c^2+4D_2\lambda}\right)}{2D_2} - \frac{c(x-ct)}{D_1}} \right) \left(\frac{e^{-\frac{(x-ct)(c+\sqrt{c^2+4D_2\lambda})}{2D_2}}}{c\lambda\left(c^2(D_1-D_2)+D_1^2\lambda\right)} \right)$$

9. $p \neq 0$, $\lambda \neq 0$ and $\delta = 0$:

$$N(x,t) = \left(\frac{p(k-c(x-ct))}{D_1}\right)^{1/p}$$

$$E(x,t) = c_1$$

$$M(x,t) = \frac{1}{2\lambda\alpha}$$
$$\left(\left(\frac{2cD_2 p}{D_1(c-\alpha)} \right)^{1/p} \mu\,(\alpha + c)\, e^{\left(\frac{(c(x-ct)-k)(\alpha-c)}{2cD_2}\right)} \Gamma\left(1 + \frac{1}{p}, \frac{(k-c(x-ct))(c-\alpha)}{2cD_2}\right) \right.$$
$$+ \left(\frac{2pcD_2}{D_1(c+\alpha)} \right)^{1/p} \mu\,(\alpha - c)\, e^{\frac{(k-c(x-ct))(c+\alpha)}{2cD_2}} \Gamma\left(1 + \frac{1}{p}, \frac{(k-c(x-ct))(c+\alpha)}{2cD_2}\right)$$
$$\left. +2\lambda\alpha \left(c_1 e^{\frac{(-c(x-ct))(c+\alpha)}{2cD_2}} + c_2 e^{\frac{-c^2(x-ct)+(\alpha)(-2k+c(x-ct))}{2cD_2}} \right) \right),$$

where $\alpha = \sqrt{c^2 + 4D_2\lambda}$.

10. $p \neq 0$, $\lambda = 0$ and $\delta = 0$:

$$N(x,t) = \left(\frac{p(k-c(x-ct))}{D_1}\right)^{1/p}$$

$$E(x,t) = c_1$$

$$M(x,t) = e^{\frac{c(x-ct)}{D_2}} \left(-\frac{D_2 \mu p e^{-\frac{k}{D_2}} \left(\frac{p(k-c(x-ct))}{D_1}\right)^{1/p} \left(\frac{c(x-ct)-k}{D_2}\right)^{-1/p} \Gamma\left(2+\frac{1}{p}, \frac{c(x-ct)-k}{D_2}\right)}{c^2(p+1)} + c_1 \right)$$

11. $p = 0$, $\lambda = 0$ and $\delta = 0$:

$$N(x,t) = c_2 e^{-\frac{c(x-ct)}{D_1}} + \frac{c_1 D_1}{c}$$

$$E(x,t) = c_1$$

$$M(x,t) = \frac{c_2 D_1^2 \mu e^{-\frac{c(x-ct)}{D_1}}}{c^2(D_1-D_2)} + \frac{c_1 D_1 \mu (x-ct)}{c^2} - \frac{c_3 D_2 e^{-\frac{c(x-ct)}{D_2}}}{c} + c_4$$

5.1.1 Exact solution for p, λ nonzeros and $\mu = 0$

Consider the solution to system (5) when $p \neq 0$, $\lambda \neq 0$, $\mu = 0$ and the biological parameters estimated in section 2.4. Particularly, analyzing equation (138) we conclude $c_2 \neq 0$ does not hold in the biological sense, then $c_2 = 0$. This is due to the fact that we are searching for a solution moving to the right over time. Furthermore, there is no tumour on the wavefront, defined as the point where $N(x,t) = 0$, which does not hold if $c_2 \neq 0$. On the other hand, equation (140) gives $\lim_{x\to\infty} E(x,t) = c_3$ and $\lim_{x\to\infty} M(x,t) = 0$, then we set $c_3 = 1$ and $c_1 = 1$.

Rewriting solution (140) considering the parameters set and $\alpha_1 = \frac{c+\sqrt{c^2+4D_2\lambda}}{2D_2}$ we have

$$
\begin{aligned}
N(x,t) &= \begin{cases} \left(\frac{p}{D_1}\right)^{\frac{1}{p}} \left(-cx + c^2 t + k + \rho e^{-\frac{\delta}{c}\left(\frac{e^{-\alpha_1(x-ct)}}{\alpha_1}\right)}\right)^{1/p}, & \text{if } x \leq x_0(t) \\ 0, & \text{if } x > x_0(t) \end{cases} \\
E(x,t) &= e^{-\frac{\delta}{c}\left(\frac{e^{-\alpha_1(x-ct)}}{\alpha_1}\right)}, \\
M(x,t) &= e^{-\alpha_1(x-ct)}.
\end{aligned}
$$

The solution $N(x,t)$ was built based on the same idea as the porous media equation [15] which represents a traveling wave – a wave that advances in time with a constant velocity and maintaining its shape – with front $x_0(t)$. In our model the solution $N(x,t)$ is a traveling wave with constant wave speed $\frac{dx_0}{dt} = c$. When haptotaxis is zero, a limit case because it is not a solution, we can calculate the front $x_0(t)$. In this case the wave front is $x_0(t) = ct + c^{-1}k$. The haptotaxis effect increases at the wave front.

In addition to that, we chose $k = 0$ since k gives a translation of the solution and $c = 0.045$ so that the tumour is within the expected range. All parameters related to the graphics for this solution were combined into Table 4.

Table 4: Values of parameters set to the graphics of the solution to system (5) when $\mu = 0, \lambda \neq 0$ and $p \neq 0$.

Parameters	D_1	D_2	ρ	δ	λ	c	c_1	c_2	c_3	k
Values	0.001	0.001	0.005	10	0.5	0.045	1	0	1	0

Figures 15, 16 and 17 show respectively MDE concentration, density of ECM and cancer cells density for $p = 1$ at $t = 5, t = 10, t = 15$ and $t = 20$. We also can see a travelling wave solution with speed $c = 0.045$.

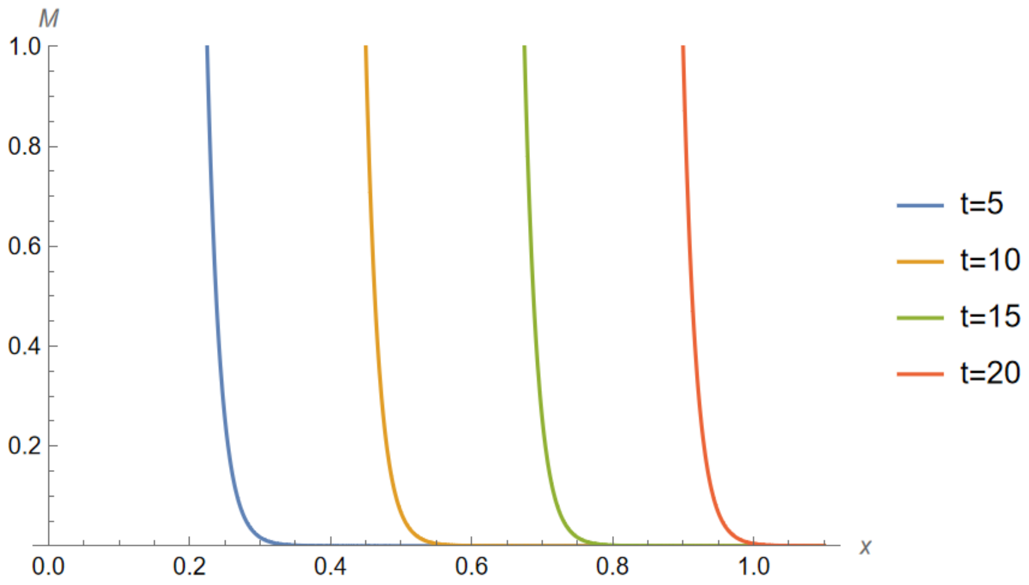


Figure 15: *MDE* concentration for $p = 1$ at $t = 5, t = 10, t = 15$ and $t = 20$ with other parameters into Table 4.

Source: the author.

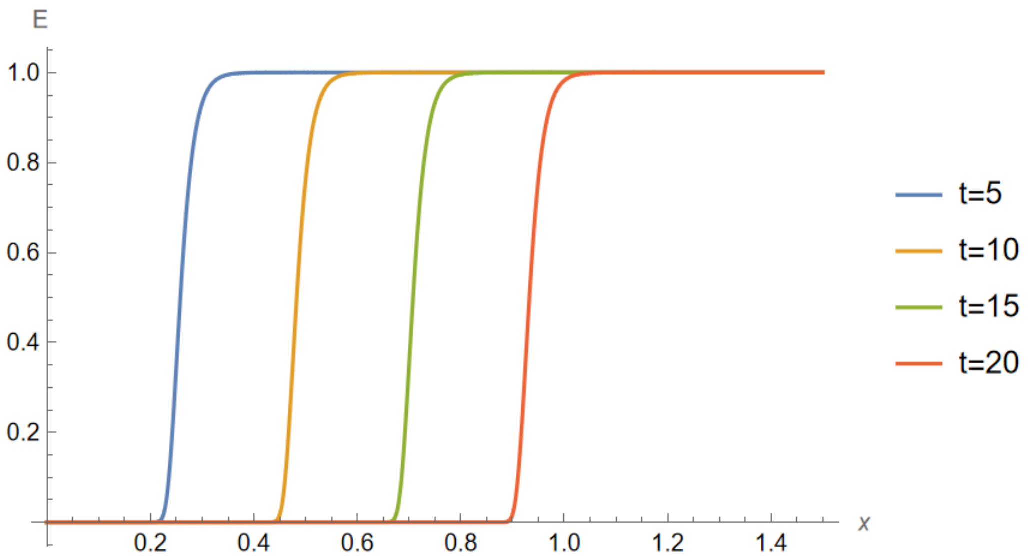


Figure 16: Density of *ECM* for $p = 1$ at $t = 5, t = 10, t = 15$ and $t = 20$ with other parameters into Table 4.

Source: the author.

Figure 15 also shows that *MDE* disseminates by diffusion D_2 and its density does not increase as time evolves due to $\mu = 0$. Moreover, the *ECM* profile presented in Figure 16 shows its degradation by the *MDE* at $t = 5, t = 10, t = 15$ and $t = 20$.

In Figure 17 we can observe an interesting phenomenon in the front of the wave: there is a cluster of cells invading tissue further than the main body of the tumour, moving to the right.

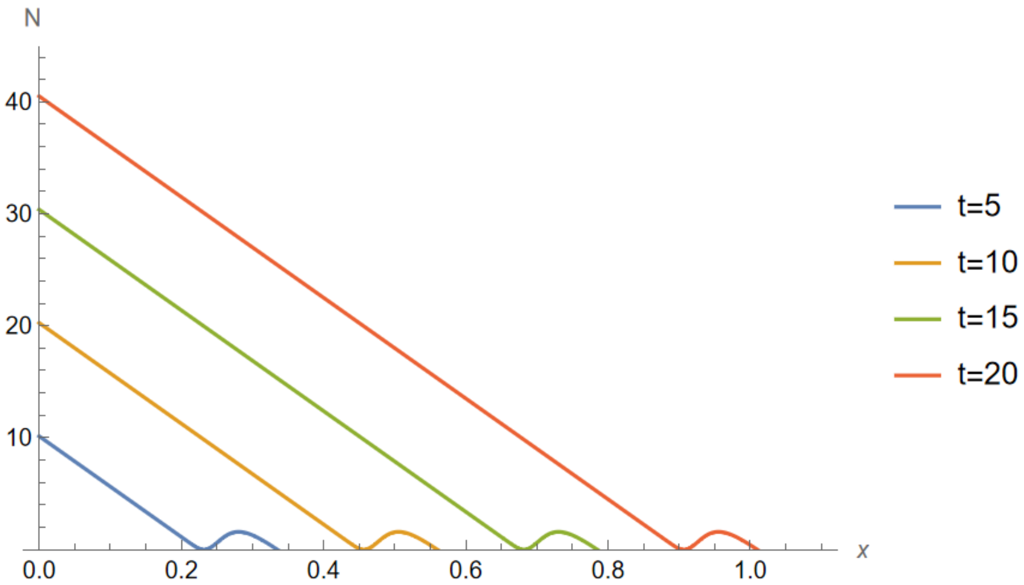


Figure 17: Cancer cells density for $p = 1$ at $t = 5, t = 10, t = 15$ and $t = 20$ with other parameters into Table 4.

Source: the author.

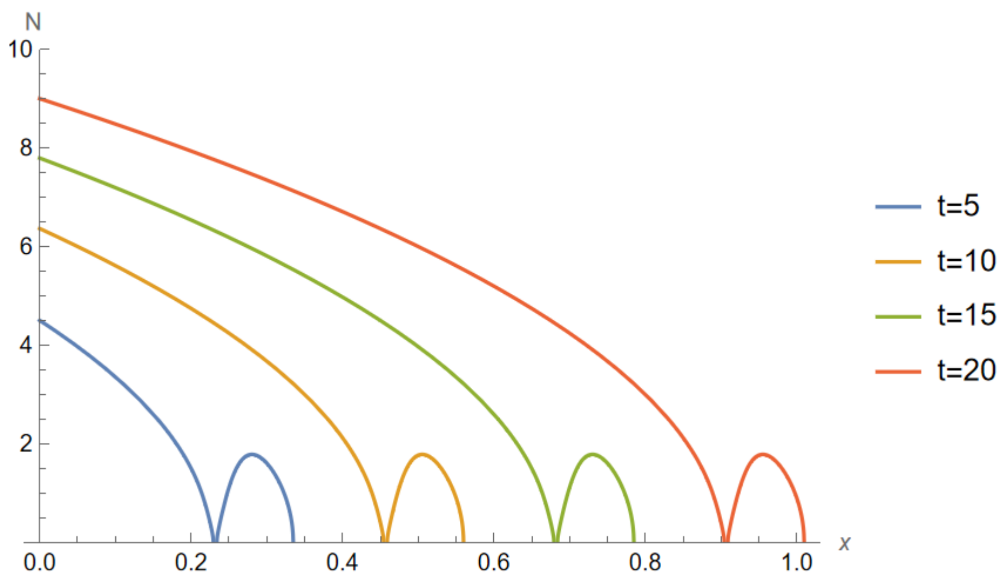


Figure 18: Cancer cells density for $p = 2$ at $t = 5, t = 10, t = 15$ and $t = 20$ with other parameters into Table 4.

Source: the author.

In order to investigate the phenomenon seen in Figure 17 we maintain parameters set into Table 4 and take $p = 2$ at $t = 5$, $t = 10$, $t = 15$ and $t = 20$, presented in Figure 18, and also $p = 1, 2, 3$ at different values of t presented into Figures 19, 20, 21 and 22.

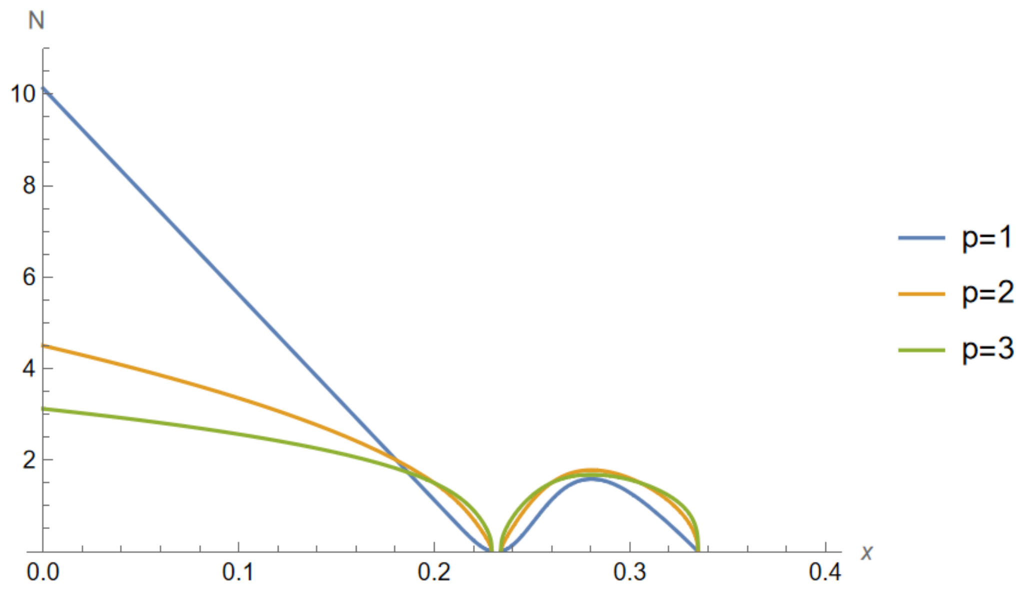


Figure 19: Cancer cells density for $p = 1, 2, 3$, at $t = 5$ with other parameters into Table 4.
Source: the author.

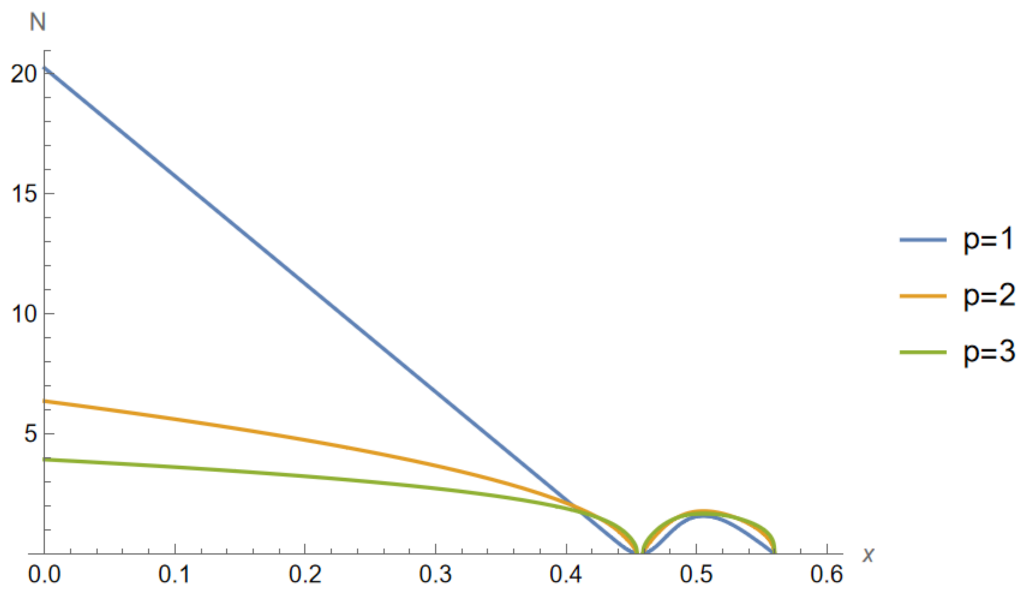


Figure 20: Cancer cells density for $p = 1, 2, 3$, at $t = 10$ with other parameters into Table 4.
Source: the author.

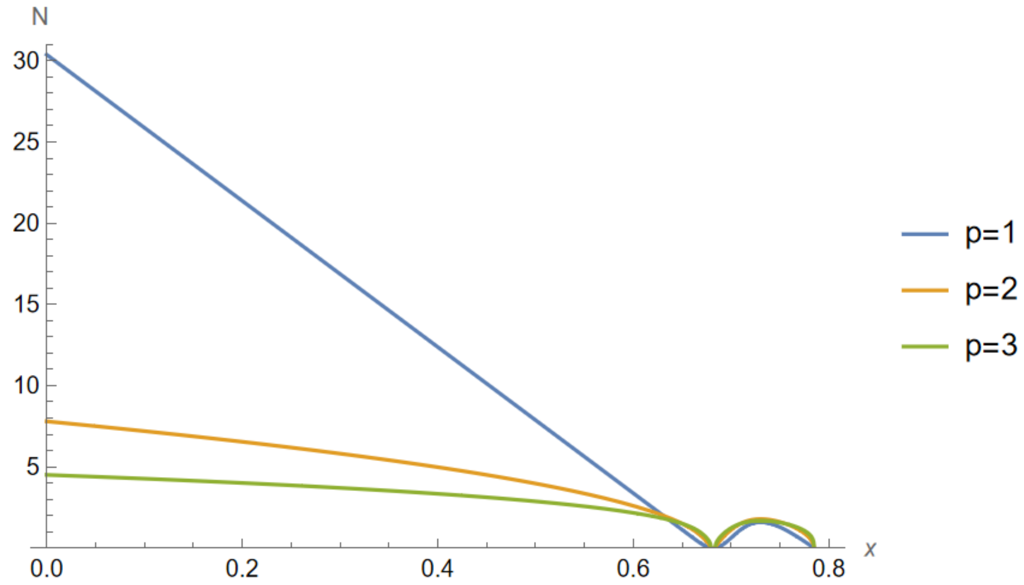


Figure 21: Cancer cells density for $p = 1, 2, 3$, at $t = 15$ with other parameters into Table 4.
Source: the author.

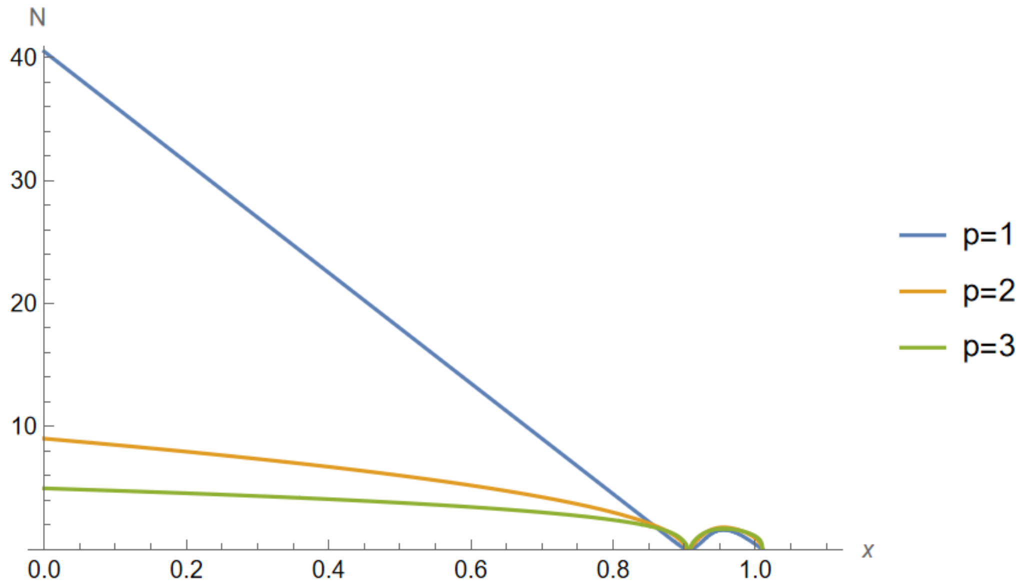


Figure 22: Cancer cells density for $p = 1, 2, 3$, at $t = 20$ with other parameters into Table 4.
Source: the author.

From these graphics we can carefully infer that this effect in cancer cells density can be associated to the dependence diffusion by tumour cells and haptotaxis, which can also be seen in Figure 23 when $p = 2, x \in [0, 2]$ and $t \in [0, 20]$.

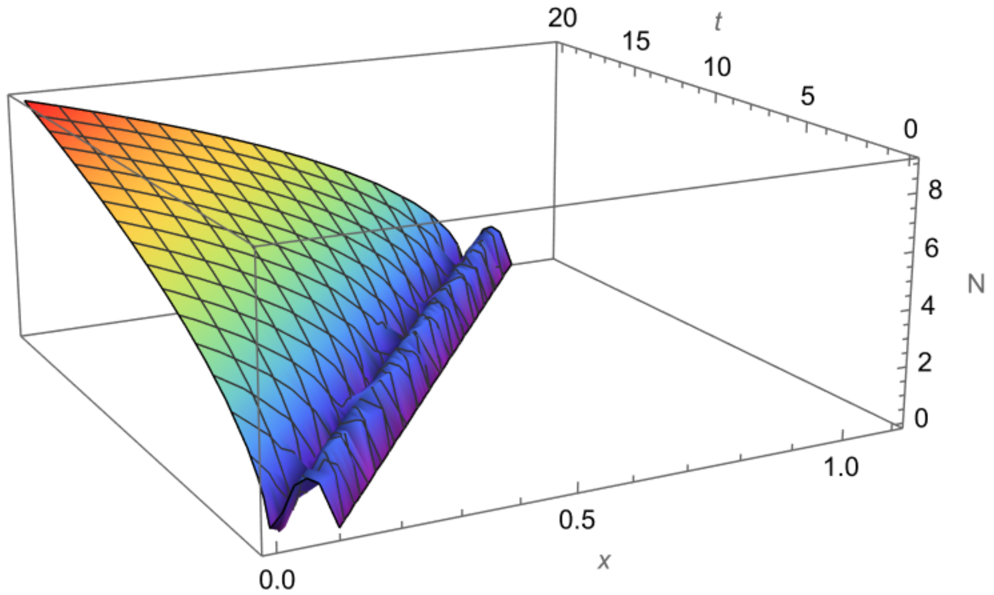


Figure 23: Cancer cells density for $p = 2, x \in [0,2]$ and $t \in [0,20]$ with other parameters into Table 4.

Source: the author.

In Figure 23 the traveling wave moves to right over time with a constant shape. Moreover, at the wavefront we observe the haptotaxis effect.

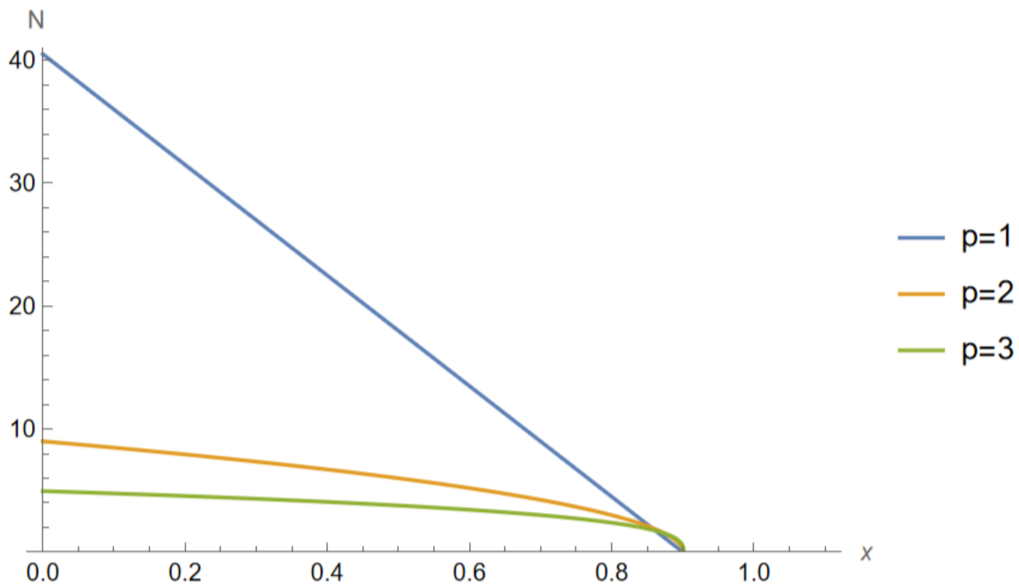


Figure 24: Cancer cells density for $p = 1, 2, 3$ at $t = 20$ with other parameters into Table 4 except ρ taken here as $\rho = 0.0025$.

Source: the author.

Analysing $N(x,t)$ for the same preceding parameters changing only $\rho = 0.0025$, i.e., half of the value of ρ tested at $t = 20$, we have Figure 24. Furthermore, we can observe that haptotaxis effect is imperceptible at this scale.

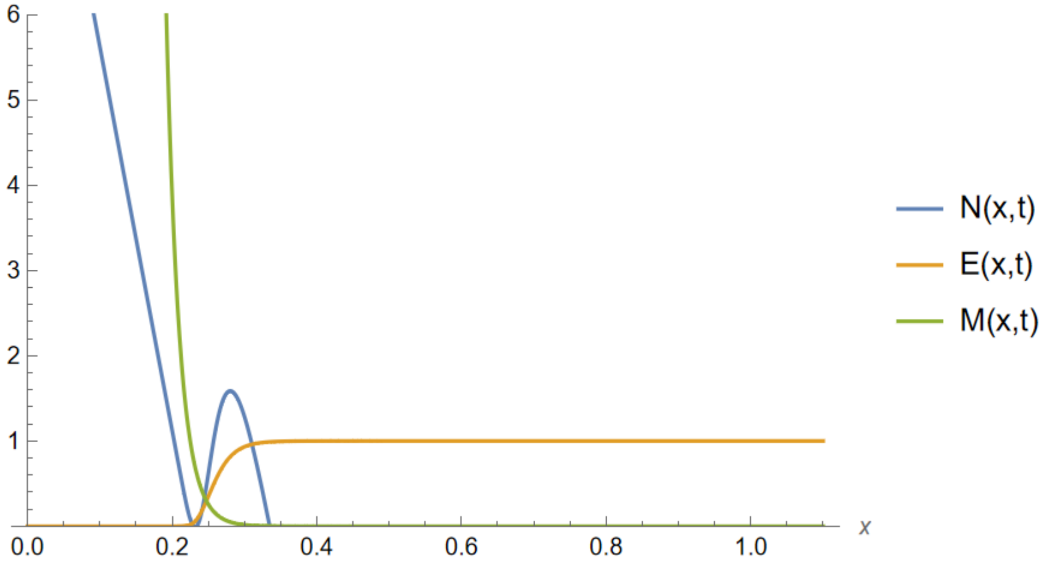


Figure 25: *MDE* concentration, cancer cells density and density of *ECM* for $p = 1$ at $t = 5$ with other parameters into Table 4.

Source: the author.

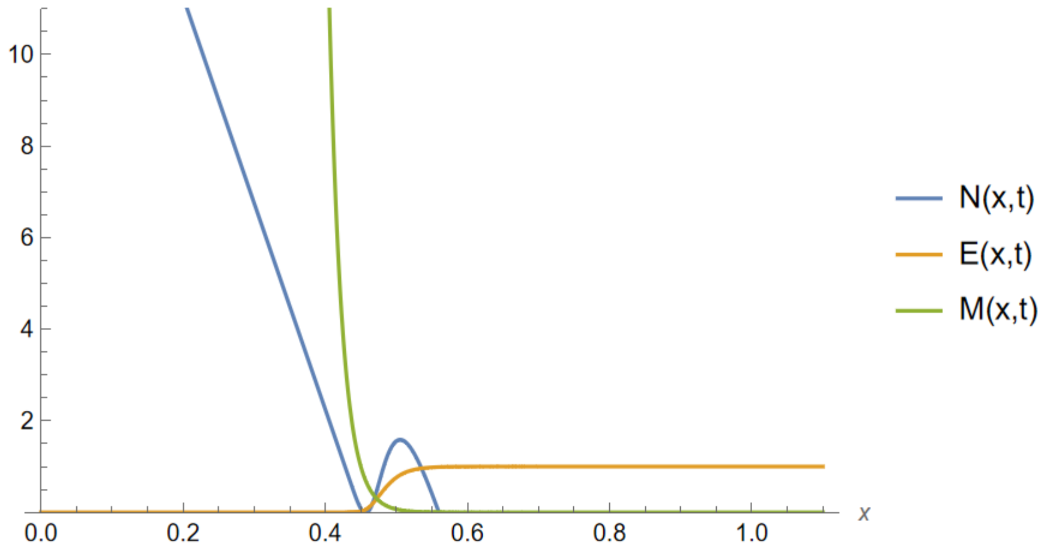


Figure 26: *MDE* concentration, cancer cells density and density of *ECM* for $p = 1$ at $t = 10$ with other parameters into Table 4.

Source: the author.

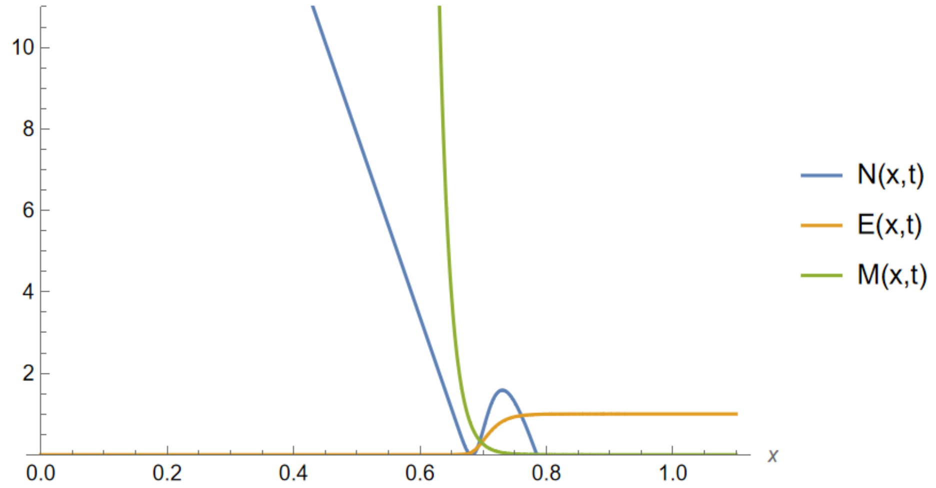


Figure 27: *MDE* concentration, cancer cells density and density of *ECM* for $p = 1$ at $t = 15$ with other parameters into Table 4.

Source: the author.

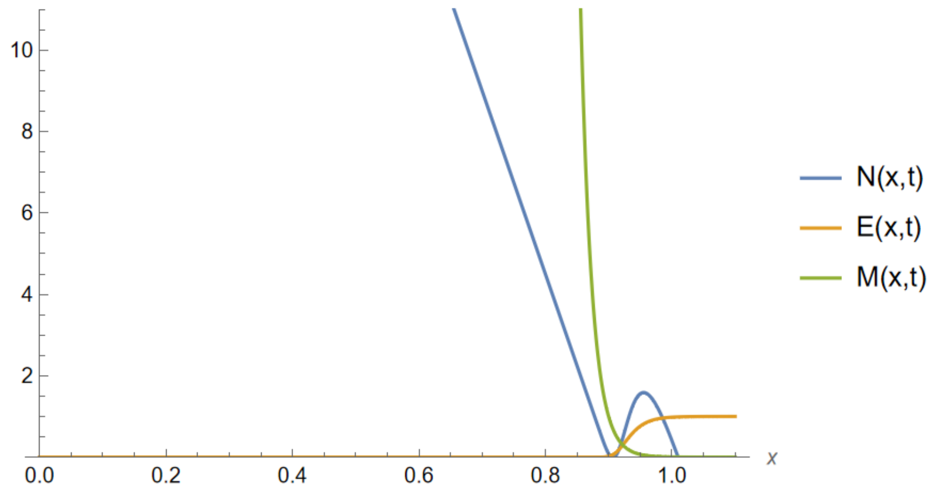


Figure 28: *MDE* concentration, cancer cells density and density of *ECM* for $p = 1$ at $t = 20$ with other parameters into Table 4.

Source: the author.

Now analysing *MDE* concentration, cancer cells density and density of *ECM* for $p = 1$ at $t = 5, 10, 15, 20$ and parameters in Table 4 in a combined way, we have Figures 25, 26, 27 and 28, respectively. Those graphics show the travelling wave solutions with wave speed $c = 0.045$ with wavefront approximately at $x = 0.334944$, $x = 0.559944$, 0.784944 and 1.00994, respectively. Comparing these results with one dimensional numerical simulations in [3] we observe a similar behaviour reinforcing the set velocity at the present work.

5.1.2 Exact solution for $p = \mu = 0$ and λ nonzero

Figures 29, 30, 31 and 32 show a comparison of *MDE* concentration, cancer cells density and density of *ECM* for $p = 0$ at $t = 5, 10, 15, 20$ and parameters in Table 4, respectively.

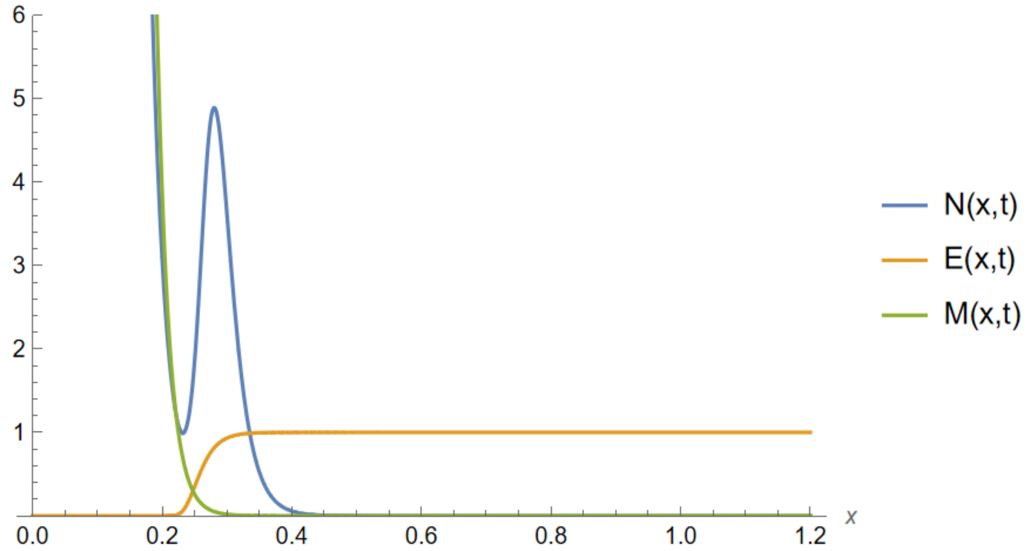


Figure 29: *MDE* concentration, cancer cells density and density of *ECM* for $p = 0$ at $t = 5$ with other parameters into Table 4.
Source: the author.

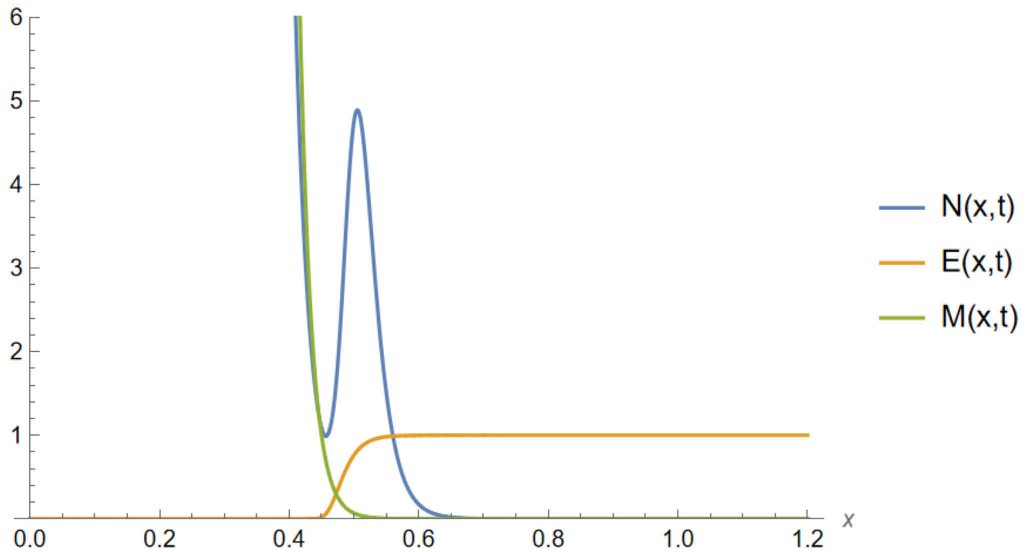


Figure 30: *MDE* concentration, cancer cells density and density of *ECM* for $p = 0$ at $t = 10$ with other parameters into Table 4.
Source: the author.

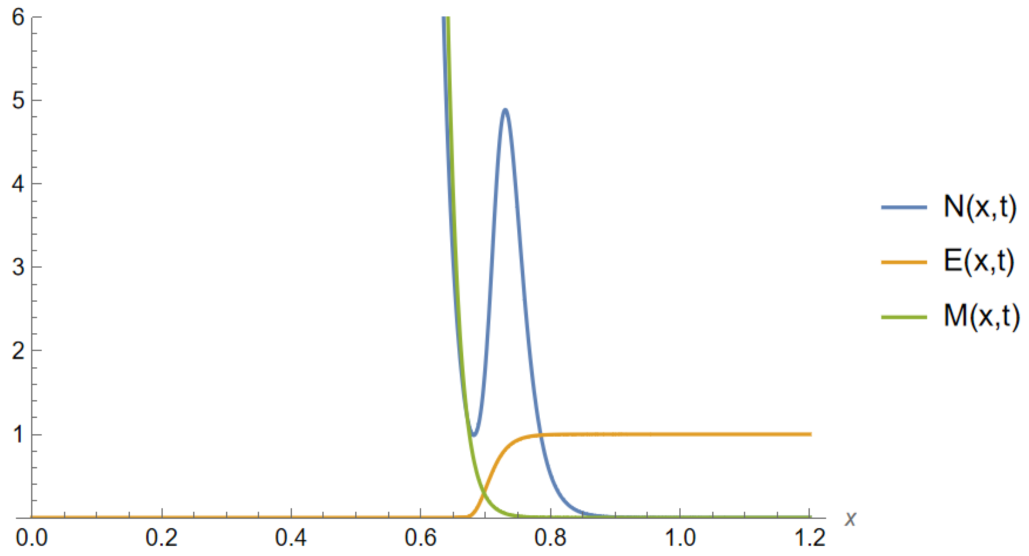


Figure 31: *MDE* concentration, cancer cells density and density of *ECM* for $p = 0$ at $t = 15$ with other parameters into Table 4.

Source: the author.

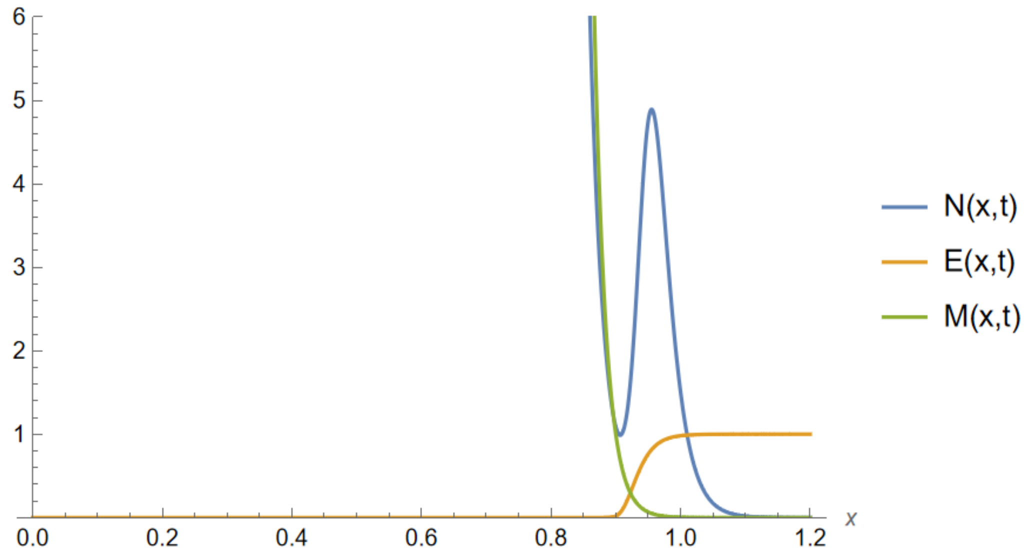


Figure 32: *MDE* concentration, cancer cells density and density of *ECM* for $p = 0$ at $t = 20$ with other parameters into Table 4.

Source: the author.

From all these graphics we can observe that the behaviour of the solution when $p = 0$ is similar when $p \neq 0$ although we obtained different analytical solutions. Thus the biological analysis here is the same as the one of the previous subsection, including the haptotaxis effect at the wavefront.

5.2 GENERATOR X_{28}

In this section, we will analyse generator $X_{28} = \frac{x}{2}\partial_x + t\partial_t - M\partial_M$.

5.2.1 Case p, ρ and δ nonzeros, and $\mu = \lambda = 0$

Generator X_{28} is valid when the system parameters are given by $p \neq 0, \rho \neq 0, \delta \neq 0, \mu = 0$ and $\lambda = 0$.

Original system (5) for this case can be written as

$$\begin{cases} N_t &= D_1(pN^{p-1}N_xN_x + N^pN_{xx}) - \rho(N_xE_x + NE_{xx}), \\ E_t &= -\delta ME, \\ M_t &= D_2M_{xx}. \end{cases} \tag{141}$$

Characteristic system associated with generator X_{28} is given by

$$\frac{dx}{\frac{x}{2}} = \frac{dt}{t} = \frac{dN}{0} = \frac{dE}{0} = \frac{dM}{-M}$$

and its associated invariants can be set by

$$w = \frac{x^2}{t}, J_1 = N, J_2 = E, J_3 = Mx^2. \tag{142}$$

Assuming $J_1 = \Phi_1(w), J_2 = \Phi_2(w), J_3 = \Phi_3(w)$, where w is as in (142), we have

$$N = \Phi_1(w), E = \Phi_2(w), M = \frac{\Phi_3(w)}{x^2}. \tag{143}$$

In order to rewrite the system (141) considering the new variables, we need to find $N_t, N_x, N_{xx}, E_t, E_x, E_{xx}, M_t, M_{xx}$ considering (143). Thus,

$$N_t = -\frac{x^2\Phi_1'}{t^2} \tag{144}$$

$$N_x = \frac{2\Phi_1'x}{t} \tag{145}$$

$$N_{xx} = \frac{4x^2\Phi_1''}{t^2} + \frac{2\Phi_1'}{t} \tag{146}$$

$$E_t = -\frac{x^2\Phi_2'}{t^2} \tag{147}$$

$$E_x = \frac{2x\Phi_2'}{t} \tag{148}$$

$$E_{xx} = \frac{4x^2\Phi_2''}{t^2} + \frac{2\Phi_2'}{t} \tag{149}$$

$$M_t = -\frac{\Phi_3'}{t^2} \tag{150}$$

$$M_x = \frac{2\Phi_3'}{tx} - \frac{2\Phi_3}{x^3} \tag{151}$$

$$M_{xx} = \frac{4\Phi_3''}{t^2} - \frac{6\Phi_3'}{tx^2} + \frac{6\Phi_3}{x^4} \tag{152}$$

Substituting $w = \frac{x^2}{t}$ and (143)-(152) into system (141), we obtain:

$$\begin{cases} -w\Phi_1' &= D_1(4pw\Phi_1^{p-1}\Phi_1'^2 + 4w\Phi_1^p\Phi_1'' + 2\Phi_1^p\Phi_1') - \rho(4w\Phi_1'\Phi_2' + 4w\Phi_1\Phi_2'' + 2\Phi_1\Phi_2'), \\ w^2\Phi_2' &= \delta\Phi_3\Phi_2, \\ -w^2\Phi_3' &= D_2(4w^2\Phi_3'' - 6w\Phi_3' + 6\Phi_3). \end{cases} \tag{153}$$

Third and second equations in (153) provide, respectively,

$$\Phi_3 = c_2 w^{3/2} e^{-\frac{w}{4D_2}}$$

and

$$\Phi_2 = c_3 e^{-2c_2\delta\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)},$$

where $\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)$ is the incomplete gamma function given by

$$\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right) = \int_{\frac{w}{4D_2}}^{\infty} t^{-\frac{1}{2}} e^{-t} dt.$$

Substituting

$$\Phi_2' = \frac{c_2 c_3 \delta e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right) - \frac{w}{4D_2}}}{\sqrt{w}}$$

and

$$\Phi_2'' = -\frac{c_2 c_3 \delta e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right) - \frac{w}{2D_2}}\left(2D_2 e^{\frac{w}{4D_2}} - 4D_2 c_2 \delta\sqrt{w} + w e^{\frac{w}{4D_2}}\right)}{4D_2 w^{3/2}}$$

into first equation of (153), we have

$$
\begin{aligned}
-w\Phi_1' &= D_1(4pw\Phi_1^{p-1}\Phi_1'^{\,2} + 4w\Phi_1^p\Phi_1'' + 2\Phi_1^p\Phi_1') \\
&-\rho 4w\Phi_1'\left(\frac{c_2c_3\delta e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)-\frac{w}{4D_2}}}{\sqrt{w}}\right) \\
&+4w\rho\Phi_1\left(\frac{c_2c_3\delta e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)-\frac{w}{2D_2}}\left(2D_2e^{\frac{w}{4D_2}}-4D_2c_2\delta\sqrt{w}+we^{\frac{w}{4D_2}}\right)}{4D_2w^{3/2}}\right) \\
&-2\rho\Phi_1\left(\frac{c_2c_3\delta e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)-\frac{w}{4D_2}}}{\sqrt{w}}\right).
\end{aligned} \tag{154}
$$

Rewriting (154), we obtain

$$
\begin{aligned}
-w\Phi_1' &= D_1(4pw\Phi_1^{p-1}\Phi_1'^{\,2} + 4w\Phi_1^p\Phi_1'' + 2\Phi_1^p\Phi_1') \\
&+\left(\frac{-4\delta c_2\Phi_1}{e^{\frac{w}{2D_2}}} + \frac{\sqrt{w}\Phi_1}{D_2e^{\frac{w}{4D_2}}} - \frac{4\sqrt{w}\Phi_1'}{e^{\frac{w}{4D_2}}}\right)\left(c_2c_3\delta\rho e^{-2\delta c_2\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)}\right).
\end{aligned} \tag{155}
$$

We have not been able to find any analytical solution to (155), i.e., $N(x,t)$. However we can analyze $E(x,t)$ and $M(x,t)$ given by $E(x,t) = c_3e^{-2c_2\delta\sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{x^2}{4D_2t}\right)}$ and $M(x,t) = c_2\dfrac{x}{t^{3/2}}e^{-\frac{x^2}{4D_2t}}$.

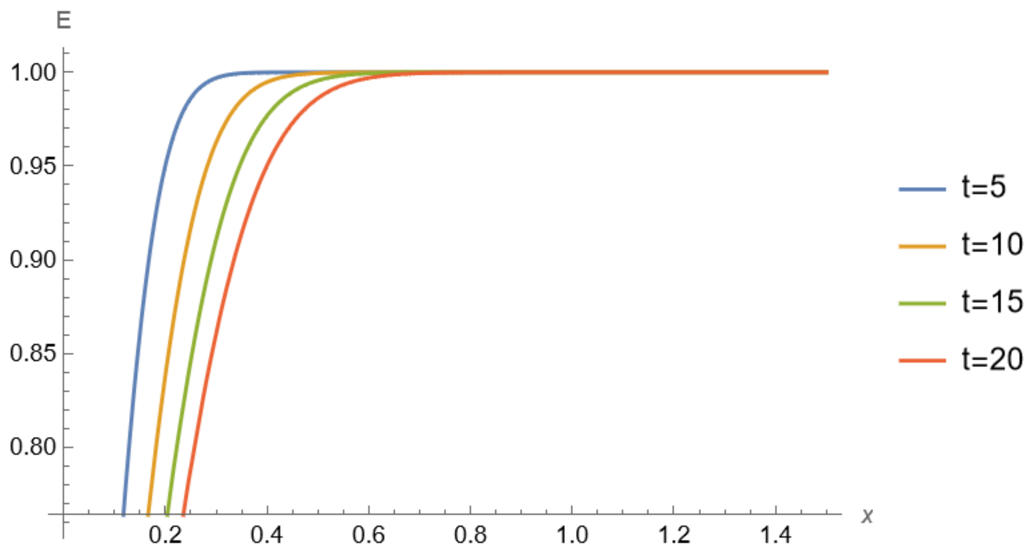


Figure 33: Density of ECM at $t=5, t=10, t=15$ and $t=20$ with $c_3=1$, $D_2=0.001$ and $\delta=10$. Source: the author.

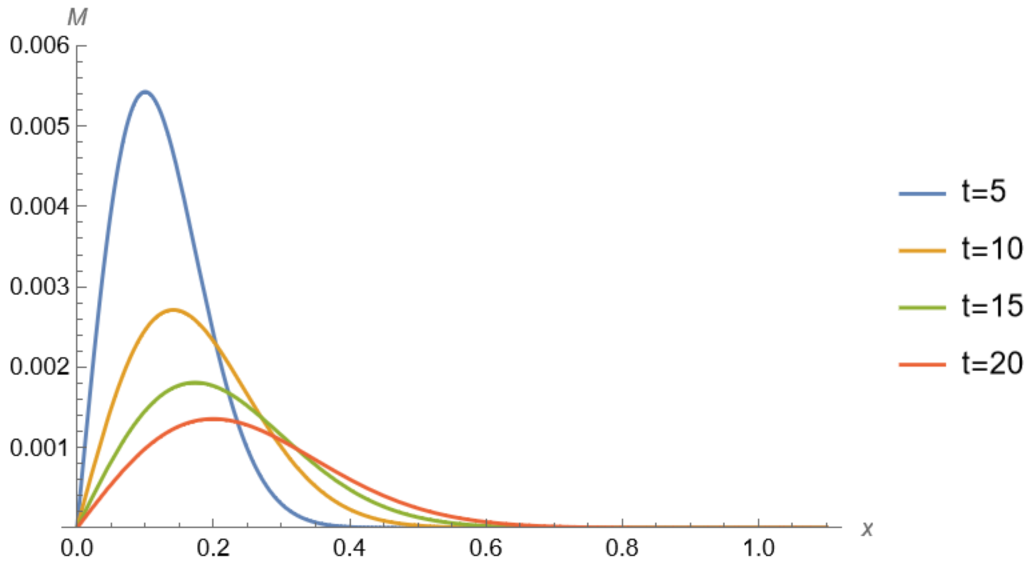


Figure 34: *MDE* concentration at $t = 5, t = 10, t = 15$ and $t = 20$ with $c_2 = 1$, $D_2 = 0.001$.
Source: the author.

So we maintain the expected scale and biological meaning, we set $c_3 = 1$ and $c_2 > 0$, $c_2 = 1$ at first.

The *ECM* profile presented in Figure 33 shows its degradation by the *MDE* at $t = 5, t = 10, t = 15$ and $t = 20$, with a slowly rate decreasing. Moreover, Figure 34 shows that *MDE* disseminates by diffusion D_2 and its density decrease as time evolves. Here, we no longer have travelling waves whose shape changes over time.

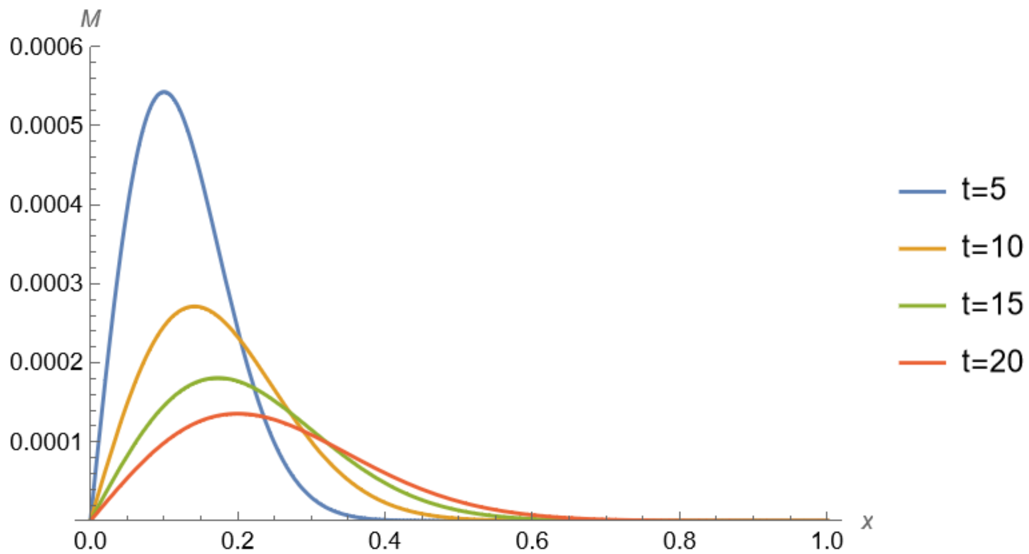


Figure 35: *MDE* concentration at $t = 5, t = 10, t = 15$ and $t = 20$ with $c_2 = 0.1$, $D_2 = 0.001$.
Source: the author.

Although the *MDE* concentration range appears to be low in this case, the result is similar to the [3, figure 4b, page 137] when some parameters were changed to test their effect on the solution. Figures 35 and 36 show that a modification in c_2 causes a change in the density scale *MDE*, with the same behavior.

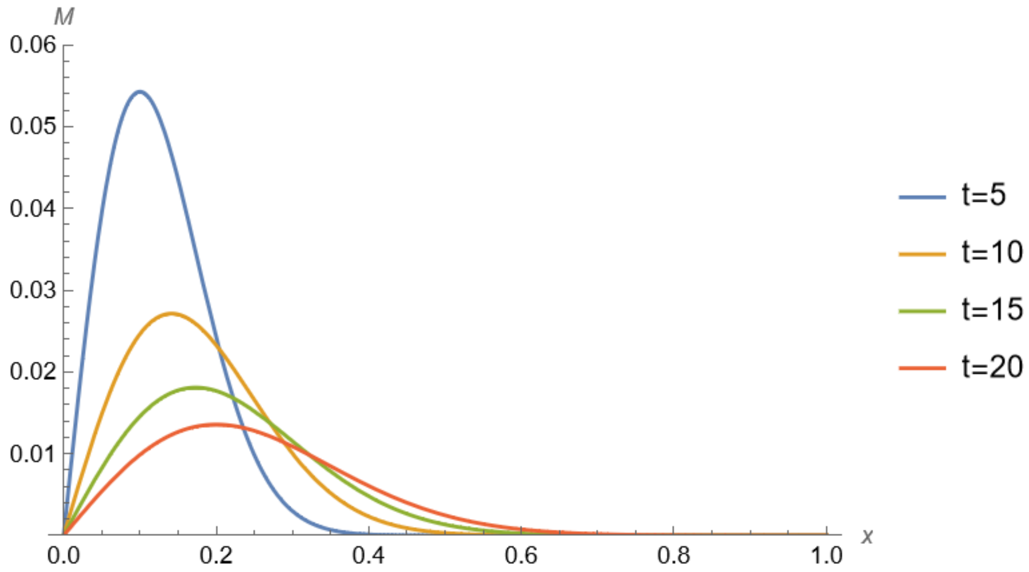


Figure 36: *MDE* concentration at $t = 5, t = 10, t = 15$ and $t = 20$ with $c_2 = 10$, $D_2 = 0.001$.
Source: the author.

Moreover, Figures 37 and 38 present similar behaviour of density of *ECM* when compared to $c_2 = 1$ profile.

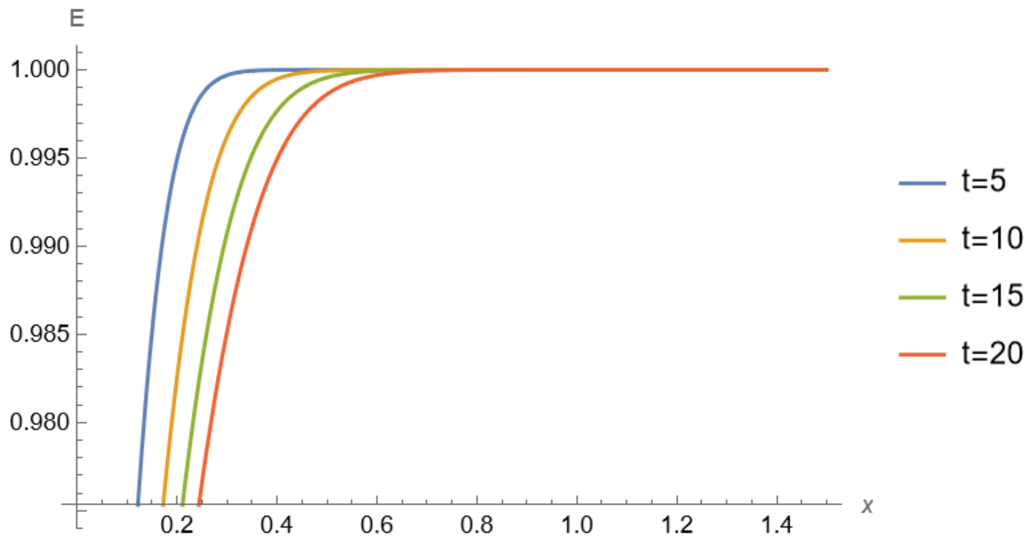


Figure 37: Density of *ECM* at $t = 5, t = 10, t = 15$ and $t = 20$ with $c_2 = 0.1$, $D_2 = 0.001$.
Source: the author.

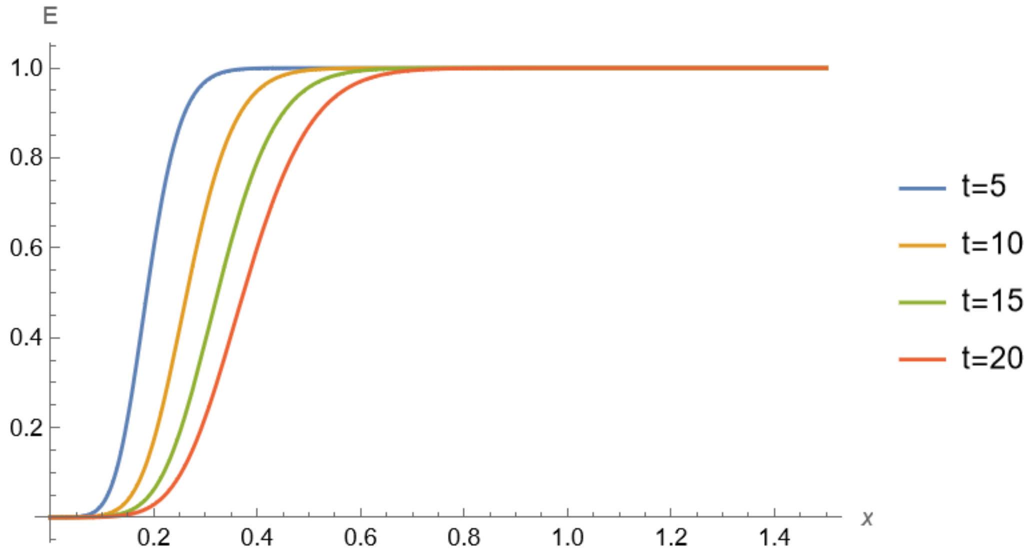


Figure 38: Density of ECM at $t = 5, t = 10, t = 15$ and $t = 20$ with $c_2 = 10$, $D_2 = 0.001$.
Source: the author.

5.2.2 Case ρ and δ nonzeros, $p = \mu = \lambda = 0$

Generator X_{28} is also valid when the system parameters are given by $p = 0, \rho \neq 0, \delta \neq 0, \mu = 0$ and $\lambda = 0$.

The original system (5) for this case can be written as

$$\begin{cases} N_t &= D_1 N_{xx} - \rho(N_x E_x + N E_{xx}), \\ E_t &= -\delta M E, \\ M_t &= D_2 M_{xx}. \end{cases}$$

Similarly to the previous case, we obtain

$$N = \Phi_1(w), E = \Phi_2(w), M = \frac{\Phi_3(w)}{x^2},$$

where $w = \frac{x^2}{t}$.

Besides that, we have

$$\Phi_2 = c_3 e^{-2c_2\delta\sqrt{D_2}\Gamma\left(\frac{1}{2}, \frac{w}{4D_2}\right)},$$
$$\Phi_3 = c_2 w^{3/2} e^{-\frac{w}{4D_2}},$$

and

$$\begin{aligned} -w\Phi_1' &= D_1(4w\Phi_1'' + 2\Phi_1') \\ &+ \left(\frac{-4\delta c_2 \Phi_1}{e^{\frac{w}{2D_2}}} + \frac{\sqrt{w}\Phi_1}{D_2 e^{\frac{w}{4D_2}}} - \frac{4\sqrt{w}\Phi_1'}{e^{\frac{w}{4D_2}}} \right) \left(c_2 c_3 \delta \rho e^{-2\delta c_2 \sqrt{D_2}\Gamma\left(\frac{1}{2}, \frac{w}{4D_2}\right)} \right), \end{aligned}$$

i.e.,

$$
\begin{aligned}
0 \;=\; & \Phi_1''(4D_1 w) \\
& +\Phi_1'\left(w + 2D_1 - \frac{4\sqrt{w}}{e^{\frac{w}{4D_2}}} c_2 c_3 \delta \rho e^{-2\delta c_2 \sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)}\right) \\
& +\Phi_1\left(\frac{-4\delta c_2}{e^{\frac{w}{2D_2}}} + \frac{\sqrt{w}}{D_2 e^{\frac{w}{4D_2}}}\right)\left(c_2 c_3 \delta \rho e^{-2\delta c_2 \sqrt{D_2}\Gamma\left(\frac{1}{2},\frac{w}{4D_2}\right)}\right).
\end{aligned}
$$

The solutions found here for $E(x,t)$ and $M(x,t)$ are the same as the previous case where $p \neq 0$. Then the biological analysis for both variables remains the same as well. Also in this case, we have been unable to express $N(x,t)$.

5.3 GENERATOR X_{27}

Section 4.2 gave us all the generators related to system (5). In this section, we will analyse generator $X_{27} = \frac{x}{2}\partial_x + t\partial_t - 2N\partial_N - M\partial_M$, which is valid when the system parameters are given by $p = 0, D_1 = D_2, \rho \neq 0, \delta \neq 0, \mu \neq 0$ and $\lambda = 0$.

Original system (5) for this case can be written as

$$
\begin{cases}
N_t = D_1 N_{xx} - \rho(N_x E_x + N E_{xx}), \\
E_t = -\delta M E, \\
M_t = D_1 M_{xx} + \mu N.
\end{cases} \tag{156}
$$

Characteristic system associated with generator X_{27} is given by

$$
\frac{dx}{\frac{x}{2}} = \frac{dt}{t} = \frac{dN}{-2N} = \frac{dE}{0} = \frac{dM}{-M}
$$

and invariants associated with this generator can be set by

$$
w = \frac{x^2}{t}, J_1 = Nx^4, J_2 = E, J_3 = Mx^2. \tag{157}
$$

Assuming $J_1 = \Phi_1(w), J_2 = \Phi_2(w), J_3 = \Phi_3(w)$, where w is as in (157), we have

$$
N = \frac{\Phi_1(w)}{x^4}, E = \Phi_2(w), M = \frac{\Phi_3(w)}{x^2}. \tag{158}
$$

In order to rewrite system (156) considering the new variables, we need to find $N_t, N_x, N_{xx}, E_t, E_x, E_{xx}, M_t, M_{xx}$ in view of (158). Thus:

$$
N_t = -\frac{\Phi_1'}{x^2 t^2} \tag{159}
$$

$$N_x = \frac{2\Phi_1'}{tx^3} - \frac{4\Phi_1}{x^5} \tag{160}$$

$$N_{xx} = \frac{4\Phi_1''}{t^2x^2} - \frac{14\Phi_1'}{tx^4} + \frac{20\Phi_1}{x^6} \tag{161}$$

$$E_t = -\frac{x^2\Phi_2'}{t^2} \tag{162}$$

$$E_x = \frac{2x\Phi_2'}{t} \tag{163}$$

$$E_{xx} = \frac{4x^2\Phi_2''}{t^2} + \frac{2\Phi_2'}{t} \tag{164}$$

$$M_t = -\frac{\Phi_3'}{t^2} \tag{165}$$

$$M_x = \frac{2\Phi_3'}{tx} - \frac{2\Phi_3}{x^3} \tag{166}$$

$$M_{xx} = \frac{4\Phi_3''}{t^2} - \frac{6\Phi_3'}{tx^2} + \frac{6\Phi_3}{x^4} \tag{167}$$

Substituting $w = \frac{x^2}{t}$ and (158)-(167) into system (156), we obtain the new system:

$$\begin{cases} -w^2\Phi_1' &= D_1(4w^2\Phi_1'' - 14w\Phi_1' + 20\Phi_1) - \rho(4w^2\Phi_1'\Phi_2' - 6w\Phi_1\Phi_2' + 4w^2\Phi_1\Phi_2''), \\ w^2\Phi_2' &= \delta\Phi_3\Phi_2, \\ -w^2\Phi_3' &= D_1(4w^2\Phi_3'' - 6w\Phi_3' + 6\Phi_3) + \mu\Phi_1. \end{cases} \tag{168}$$

In order to solve system (168), we assume

$$\Phi_1 = f(w)e^{\frac{-w}{4D_1}}, \Phi_2 = 0, \Phi_3 = g(w)e^{\frac{-w}{4D_1}}, \tag{169}$$

which implies $N = \dfrac{f(w)e^{\frac{-w}{4D_1}}}{x^4}$, $E = 0$ and $M = \dfrac{g(w)e^{\frac{-w}{4D_1}}}{x^2}$.

Thus, we have the following possibilities for functions $f(w)$ and $g(w)$:

$$\begin{aligned} f(w) &= w^{\frac{5}{2}}c_1(-6D_1+w) \\ &+ \frac{c_2w^2\left(8D_1^{3/2}e^{\frac{w}{4D_1}}+\sqrt{\pi}w^{3/2}\text{ erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-6\sqrt{\pi}D_1\sqrt{w}\text{ erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}we^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}} \\ g(w) &= c_1\mu w^{5/2}+c_3w^{3/2} \\ &+ \frac{c_2\mu w^{3/2}\left(-2\sqrt{\pi}D_1\text{ erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)+\sqrt{\pi}w\text{ erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}\sqrt{w}e^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}}, \end{aligned} \tag{170}$$

where c_1, c_2 and c_3 are integration constants, and $\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)$ is the imaginary error function defined by

$$\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right) = -\text{erf}\left(\frac{i\sqrt{w}}{2\sqrt{D_1}}\right) = -\frac{2}{\pi}\int_0^{\frac{i\sqrt{w}}{2\sqrt{D_1}}} t^{-\frac{1}{2}}e^{-t^2}dt.$$

So, one solution for system (168) is given by

$$\begin{aligned} N(x,t) &= \frac{e^{-\frac{x^2}{4D_1t}}}{x^4}\left(c_1\left(\frac{x^2}{t}\right)^{5/2}\left(-6D_1+\frac{x^2}{t}\right)+\frac{c_2x^4}{24D_1^{5/2}t^2}8D_1^{3/2}e^{\frac{x^2}{4D_1t}}\right. \\ &\left.+\frac{c_2x^4}{24D_1^{5/2}t^2}\left(\sqrt{\pi}\left(\frac{x^2}{t}\right)^{3/2}\text{ erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-6\sqrt{\pi}D_1\sqrt{\frac{x^2}{t}}\text{ erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-2\sqrt{D_1}\frac{x^2}{t}e^{\frac{x^2}{4D_1t}}\right)\right), \\ E(x,t) &= 0, \\ M(x,t) &= \frac{e^{-\frac{x^2}{4D_1t}}}{x^2}\left(c_1\mu\left(\frac{x^2}{t}\right)^{5/2}+c_3\left(\frac{x^2}{t}\right)^{3/2}\right. \\ &\left.+\frac{c_2\mu\left(\frac{x^2}{t}\right)^{3/2}}{24D_1^{5/2}}\left(-2\sqrt{\pi}D_1\text{ erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)+\sqrt{\pi}\frac{x^2}{t}\text{ erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-2\sqrt{D_1}\sqrt{\frac{x^2}{t}}e^{\frac{x^2}{4D_1t}}\right)\right). \end{aligned}$$

When we assumed $\Phi_2 = 0$, we have forced cancer cells density and density of degrading enzymes without haptotaxis. So, we make use of Perturbation Theory [24], as well as in [9]. Roughly speaking, this theory allows us to obtain approximate solutions to problems involving a very small parameter.

We are assuming $\rho \ll 1$ since we set $\rho = 0.005$ to graph some solutions along this chapter. Taking $\rho \mapsto \rho_0$ and also Φ_1, Φ_2 and Φ_3 as the targeted solutions approximated by a truncated Taylor series in first-order terms, we have $\Phi_i = \Phi_{i0} + \rho_0\Phi_{i1} + O(\rho_0^2)$, $i = 1, 2, 3$, where Φ_{i0} are the ones in (169) with (170). We are assuming that the dependence of the solution with respect to ρ_0 is sufficiently smooth.

So, rewriting (168) we have

$$\begin{cases} -w^2(\Phi_{10}+\rho_0\Phi_{11})' &= D_1(4w^2(\Phi_{10}+\rho_0\Phi_{11})'' - 14w(\Phi_{10}+\rho_0\Phi_{11})' + 20(\Phi_{10}+\rho_0\Phi_{11})) \\ & -\rho_0(4w^2(\Phi_{10}+\rho_0\Phi_{11})'(\Phi_{20}+\rho_0\Phi_{21})' \\ & -6w(\Phi_{10}+\rho_0\Phi_{11})(\Phi_{20}+\rho_0\Phi_{21})' \\ & +4w^2(\Phi_{10}+\rho_0\Phi_{11})(\Phi_{20}+\rho_0\Phi_{21})''), \\ w^2(\Phi_{20}+\rho_0\Phi_{21})' &= \delta(\Phi_{30}+\rho_0\Phi_{31})(\Phi_{20}+\rho_0\Phi_{21}), \\ -w^2(\Phi_{30}+\rho_0\Phi_{31})' &= D_1(4w^2(\Phi_{30}+\rho_0\Phi_{31})'' - 6w(\Phi_{30}+\rho_0\Phi_{31})' + 6(\Phi_{30}+\rho_0\Phi_{31})) \\ & +\mu(\Phi_{10}+\rho_0\Phi_{11}), \end{cases}$$

i.e.,

$$\begin{cases} -w^2\Phi'_{10} - w^2\rho_0\Phi'_{11} &= D_1(4w^2\Phi''_{10} + 4w^2\rho_0\Phi''_{11} - 14w\Phi'_{10} - 14w\rho_0\Phi'_{11} + 20\Phi_{10} + 20\rho_0\Phi_{11}) \\ & -\rho_0(4w^2\Phi'_{10}\Phi'_{20} + 4w^2\rho_0\Phi'_{11}\Phi'_{20} + 4w^2\rho_0\Phi'_{10}\Phi'_{21} + 4w^2\rho_0^2\Phi'_{11}\Phi'_{21} \\ & -6w\Phi_{10}\Phi'_{20} - 6w\rho_0\Phi_{11}\Phi'_{20} - 6w\rho_0\Phi_{10}\Phi'_{21} - 6w\rho_0^2\Phi_{11}\Phi'_{21} \\ & +4w^2\Phi_{10}\Phi''_{20} + 4w^2\rho_0\Phi_{11}\Phi''_{20} + 4w^2\rho_0\Phi_{10}\Phi''_{21} + 4w^2\rho_0^2\Phi_{11}\Phi''_{21}), \\ w^2\Phi'_{20} + w^2\rho_0\Phi'_{21} &= \delta(\Phi_{30}\Phi_{20} + \rho_0\Phi_{31}\Phi_{20} + \rho_0\Phi_{30}\Phi_{21} + \rho_0^2\Phi_{21}\Phi_{31}), \\ -w^2\Phi'_{30} - w^2\rho_0\Phi'_{31} &= D_1(4w^2\Phi''_{30} + 4w^2\rho_0\Phi''_{31} - 6w\Phi'_{30} - 6w\rho_0\Phi'_{31} + 6\Phi_{30} + 6\rho_0\Phi_{31}) \\ & +\mu\Phi_{10} + \mu\rho_0\Phi_{11}. \end{cases}$$

Then, disregarding $O(\rho_0^2)$ terms we obtain both systems (171) and (172):

$$O(1): \begin{cases} -w^2\Phi'_{10} &= D_1(4w^2\Phi''_{10} - 14w\Phi'_{10} + 20\Phi_{10}), \\ w^2\Phi'_{20} &= \delta\Phi_{30}\Phi_{20}, \\ -w^2\Phi'_{30} &= D_1(4w^2\Phi''_{30} - 6w\Phi'_{30} + 6\Phi_{30}) + \mu\Phi_{10}, \end{cases} \tag{171}$$

$$O(\rho_0): \begin{cases} -w^2\Phi'_{11} &= D_1(4w^2\Phi''_{11} - 14w\Phi'_{11} + 20\Phi_{11}) \\ & -(4w^2\Phi'_{10}\Phi'_{20} - 6w\Phi_{10}\Phi'_{20} + 4w^2\Phi_{10}\Phi''_{20}), \\ w^2\Phi'_{21} &= \delta(\Phi_{31}\Phi_{20} + \Phi_{30}\Phi_{21}), \\ -w^2\Phi'_{31} &= D_1(4w^2\Phi''_{31} - 6w\Phi'_{31} + 6\Phi_{31}) + \mu\Phi_{11}. \end{cases} \tag{172}$$

Thus,

$$\begin{aligned}
\Phi_{11}=\Phi_{10} &= w^{\frac{5}{2}}c_1e^{\frac{-w}{4D_1}}(-6D_1+w)\\
&+e^{\frac{-w}{4D_1}}\left(\frac{c_2w^2\left(8D_1^{3/2}e^{\frac{w}{4D_1}}+\sqrt{\pi}w^{3/2}\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-6\sqrt{\pi}D_1\sqrt{w}\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}we^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}}\right),\\
\Phi_{21} &= c_4e^{\delta\left(2\sqrt{D_1\pi}(2D_1\mu c_1+c_3)\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-\frac{1}{6D_1^{3/2}}e^{-\frac{w}{4D_1}}\sqrt{w}\mu\left(24D_1^{5/2}c_1+\sqrt{\pi}c_2\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)\right)\right)},\\
\Phi_{31}=\Phi_{30} &= e^{\frac{-w}{4D_1}}(c_1\mu w^{5/2}+c_3w^{3/2})\\
&+e^{\frac{-w}{4D_1}}\left(\frac{c_2\mu w^{3/2}\left(-2\sqrt{\pi}D_1\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)+\sqrt{\pi}w\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}\sqrt{w}e^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}}\right).
\end{aligned}$$

Since $\rho\mapsto\rho_0$ and $\Phi_i\approx\Phi_{i0}+\rho_0\Phi_{i1}$, $i=1,2,3$, then we have

$$\begin{aligned}
\Phi_1 &= (1+\rho)w^{\frac{5}{2}}c_1e^{\frac{-w}{4D_1}}(-6D_1+w)\\
&+(1+\rho)e^{\frac{-w}{4D_1}}\left(\frac{c_2w^2\left(8D_1^{3/2}e^{\frac{w}{4D_1}}+\sqrt{\pi}w^{3/2}\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-6\sqrt{\pi}D_1\sqrt{w}\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}we^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}}\right),\\
\Phi_2 &= c_4\rho e^{\delta\left(2\sqrt{D_1\pi}(2D_1\mu c_1+c_3)\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-\frac{1}{6D_1^{3/2}}e^{-\frac{w}{4D_1}}\sqrt{w}\mu\left(24D_1^{5/2}c_1+\sqrt{\pi}c_2\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)\right)\right)},\\
\Phi_3 &= (1+\rho)e^{\frac{-w}{4D_1}}(c_1\mu w^{5/2}+c_3w^{3/2})\\
&+(1+\rho)e^{\frac{-w}{4D_1}}\left(\frac{c_2\mu w^{3/2}\left(-2\sqrt{\pi}D_1\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)+\sqrt{\pi}w\text{erfi}\left(\frac{\sqrt{w}}{2\sqrt{D_1}}\right)-2\sqrt{D_1}\sqrt{w}e^{\frac{w}{4D_1}}\right)}{24D_1^{5/2}}\right).
\end{aligned}$$

Therefore, the approximated solution is given by:

$$\begin{aligned}
&N(x,t)=(1+\rho)\frac{e^{-\frac{x^2}{4D_1t}}}{x^4}\left(c_1\left(\frac{x^2}{t}\right)^{5/2}\left(-6D_1+\frac{x^2}{t}\right)+\frac{c_2x^4}{24D_1^{5/2}t^2}8D_1^{3/2}e^{\frac{x^2}{4D_1t}}\right.\\
&\left.+\frac{c_2x^4}{24D_1^{5/2}t^2}\left(\sqrt{\pi}\left(\frac{x^2}{t}\right)^{3/2}\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-6\sqrt{\pi}D_1\sqrt{\frac{x^2}{t}}\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-2\sqrt{D_1}\frac{x^2}{t}e^{\frac{x^2}{4D_1t}}\right)\right),\\
&E(x,t)=c_4\rho e^{\delta\left(2\sqrt{D_1\pi}(2D_1\mu c_1+c_3)\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-\frac{1}{6D_1^{3/2}}e^{-\frac{x^2}{4D_1t}}\sqrt{\frac{x^2}{t}}\mu\left(24D_1^{5/2}c_1+\sqrt{\pi}c_2\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)\right)\right)},\\
&M(x,t)=(1+\rho)\frac{e^{-\frac{x^2}{4D_1t}}}{x^2}\left(c_1\mu\left(\frac{x^2}{t}\right)^{5/2}+c_3\left(\frac{x^2}{t}\right)^{3/2}\right.\\
&\left.+\frac{c_2\mu\left(\frac{x^2}{t}\right)^{3/2}}{24D_1^{5/2}}\left(-2\sqrt{\pi}D_1\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)+\sqrt{\pi}\frac{x^2}{t}\text{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{D_1t}}\right)-2\sqrt{D_1}\sqrt{\frac{x^2}{t}}e^{\frac{x^2}{4D_1t}}\right)\right).
\end{aligned}$$

The biological analysis for this case will remain as future work.

6 CONCLUDING REMARKS AND FUTURE PERSPECTIVES

In this thesis we apply Lie symmetries to the model (5) that describes the interaction among cancer cell density, extracellular matrix density and concentration of a generic matrix-degrading enzyme. This model is a generalization of the continuous 1-dimensional one proposed in [3], where it was solved numerically and presented simulations assuming constant diffusion and also diffusion D directly proportional to MDE concentration. In [16] it was considered a spatially dependence diffusion of tumour cells related to brain cancer whilst [22] used a dependence of tumour cells. In view of these, here we studied the model analytically, considering diffusion as a constant, but also non-constant with a wider dependence of cancer cells density. Indeed, [3] points out that models like this look very similar to histological observations, especially when a heterogeneous ECM is introduced into them.

Using Lie's theory we carried out a complete group classification of the Lie point symmetries of the system presented into Tables (2) - (3) and then found analytical solutions to the system (5). Therefore, the method is consistent for finding solutions to a system of partial differential equations that model tumor invasion as highlighted in [8] and [9], where was published also analytical solutions but to a similar model considering constant diffusion and (x, y, z) as spatial variables.

Through the linear combination of the infinitesimal generators X_1 and X_2 we obtained 11 particular solutions for the system (5) and analyzed their biological consistency completely in 2 of these cases: $\mu = 0, \lambda \neq 0, p \neq 0$ (non-constant diffusion) and also $\mu = 0, \lambda \neq 0, p = 0$ (constant diffusion).

For the first case we were able to set some constants such as $c_2 = 0$, $c_1 = 1$ and $c = 0.045$, to maintain the biological sense of the solution. We conclude that $N(x, t)$ is a traveling wave solution with constant wave speed $c = 0.045$ and that the haptotaxis effect increases the wave front. Recent works have shown that cancer cells movement is also driven by a haptotactic response to ECM gradients, *in vivo* and *in vitro* situations [10, 14, 20, 21].

While the *ECM* profile indicates its degradation by the *MDE*, cancer cells invade tissue toward the right and break into 2 clusters when the haptotatic parameter ρ is set in 0.005. The division in clusters slowly disappears as ρ decreases, showing clearly its influence on this phenomenon. If a group of cells behaves similarly breaking away from the main body of a tumour, the metastatic cascade has the potential to be initiated: these cells can reach the vascular stage with a blood supply of their own.

From a medical point of view, the findings are of significant importance. In case of a resection surgery of the primary tumour, the smaller cluster of cells may escape the surgeon's scrutiny and lead to a possible recurrence. Moreover, the speed of invasion of cancer cells can also be a relevant factor for cancer treatment and care.

Furthermore, *MDE* disseminates by diffusion D_2 and its density does not increase as time evolves due to parameter $\mu = 0$. The case $\mu = 0, \lambda \neq 0, p = 0$ corroborates the findings of the first one.

In order to graph those exact solutions, the parameters used at the present work were essentially based on works [2], [3] and [9] and are summarized into Table 4. As far as we know, those are the only analytical solutions for system (5) where diffusion is non-constant. The main advantage of finding exact solutions lies in the fact that numerical methods often require one to be aware of more details of the problem in order to make a solution work correctly. We can also easily incorporate probabilistic factors into analytical solutions compared to numerical ones.

Solutions related to other infinitesimal generators are still under analysis. In section 5.2 we present partial solutions for the cases $p \neq 0, \rho \neq 0, \delta \neq 0, \mu = \lambda = 0$, and $\rho \neq 0, \delta \neq 0, p = \mu = \lambda = 0$ using generator X_{28}. Both cases show a slow decrease in the degradation rate of *ECM* by *MDE* over time and show that *MDE* spreads by diffusion D_2, which density also decreases over time. In addition to the parameters set as $\rho = 0.005, \delta = 10, \mu = \lambda = 0, D_1 = 0.001$, we analyzed different values for the constant c_1 to see its effect on the solution found, since for $c_1 = 1$ the enzyme density was low compared to previous results. From this we conclude that constant c_1 changes the range of $E(x,t)$ and $M(x,t)$ without modifying the solution behaviour. The solution regarding tumour cells remains for future work.

In section 5.3 we present a complete mathematical solution for the case $p = 0, D_1 = D_2, \rho \neq 0, \delta \neq 0, \mu \neq 0$ and $\lambda = 0$ without biological considerations, which demands a careful analysis that we also suggest for future research. In this case, due to the difficulties in finding exact solutions, we have forced $E(x,t)$ as zero and then used perturbation theory to find approximate solutions describing the phenomenon in a complete form. As a result, we obtain approximated solutions to system (5).

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