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Some aspects of introductory continuous logic

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Some aspects of introductory continuous logic

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DEDICATION

I dedicate my dissertation work to my family and many friends. A special feeling of gratitude to my loving grandparents, Guillermo and Esperanza whose words of encouragement and push for tenacity ring in my ears. My parents Edith and Edgar who always gave me all the support that I required. My siblings, cousins and aunts that have never left my side and are very special.

I also dedicate this dissertation to my many friends who have supported me throughout the process. I will always appreciate all they have done.

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The essence of mathematics lies in its freedom .

— *Georg Cantor*

ABSTRACT

We study metric structures by examining their model-theoretic properties under the view of continuous logic. Also, we compare three of those structures by ultraproduct techniques. In particular, we give characteristics of Urysohn's space among separable metric spaces.

Keywords: Metric structures, model theory, continuous logic, ultraproducts.

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1 INTRODUCTION

This thesis explains what continuous logic is, and presents some results and examples.

Continuous logic is a system of logic suitable for structures embodied with a notion of distance, like Banach spaces and probability algebras. It is derived simply from the two-valued classical logic, and it shares many things with the latter, including an ultraproduct construction, a compactness theorem, and a form of Löwenheim-Skolem theorem.

So to get a feeling of it, consider the following axiom, which is part of the theory of probability algebras:

$$\sup_x \sup_y (\mu(x) \dot{-} \mu(x \cup y)) = 0$$

Here $\dot{-}$ is a connective defined thus:

$$\alpha \dot{-} \beta = \begin{cases} 0 & \text{if } \alpha \leq \beta \\ \alpha - \beta & \text{if } \alpha > \beta \end{cases}$$

Therefore the axiom means that, for every events x, y in the probability space, we have $\mu(x) \leq \mu(x \cup y)$. We see that μ is a $[0,1]$-valued unary relation symbol, $\cup$ is a binary operator, and sup works as a quantifier. We also see that 0 poses as the single truth value, and our text will make clear that continuous logic is a logic of approximations suitable for the usual ϵ–δ reasonings.

Continuous logic has its origins in the book *Continuous Model Theory* by Chang and Keisler [6], which was published in 1966. The subject stayed quiet for some decades, while there was development in the seventies the theory of Banach spaces with the use of nonstandard analysis or ultraproducts, which are tools from classical logic ([9] and [8]). Eventually Henson, Ben-Yacoov, Berenstein and Usvyatsov developed the adequate logical frame in [2]. Our own interest in the subject arose when we found some works of Iovino [11] applying continuous logic to functional analysis.

Our thesis has the following chapters:

In chapter 2 we review the basic concepts of classical first-order logic: languages, structures, formulas and theories, satisfaction and models. We present the ultraproduct construction, and Łoś's Theorem as a standard way to build new models and prove the Compactness Theorem. In this way, we will be able to understand better the ultraproduct in continuous logic. See [1], [10] and [15] for an extended theory.

In chapter 3 we define the basic concepts of continuous logic in the framework of metric spaces, but using classical logic as general guide. We will see that some proofs and constructions require additional detail on setting up a language, which amounts to the modulus of uniform continuity and the lack of a negation connective. We also carefully define the ultraproduct of metric structures, and compare it with the classical one when used with discrete structures.

In chapter 4 we present the construction of ultraproducts of Banach spaces, which are a theoretical tool detached from its logical counterpart. After that, we compare this ultraproduct with the constructed one in continuous logic.

In chapter 5 we study the Urysohn space by explaining some of its model-theoretic properties. The Urysohn space is a universal, ω-homogeneous metric space, which means that it is rich both in terms of subspaces, and of automorphisms. The properties that we will discuss will enrich the comparison of classical and continuous logics.

2 BASIC CONCEPTS

We now begin setting up a framework for first-order logic (also called predicate logic). Thus, the purpose of this section is mainly to fix the notation and to set the context for the remaining ones.

2.1 LANGUAGES AND STRUCTURES

Roughly speaking, we use languages to describe mathematical structures. By a structure, we mean a set equipped with a collection of distinguished functions, relations and elements.

Definition. By a *language*, we mean a disjoint union of the following sets:

(i) a set $\mathcal{R}$ of *relation symbols* and positive integers n_R for each $R \in \mathcal{R}$ (n_R tell us that R is an n_R-ary relation);

(ii) a set $\mathcal{F}$ of *function symbols* and positive integers n_f for each $f \in \mathcal{F}$ (n_f tell us that f is a function of n_f variables);

(iii) a set $\mathcal{C}$ of *constant symbols*.

From now on, let L denote a language.

Definition. A *structure* $\mathcal{A}$ *for* L consists in:

(i) a non-empty set A called the *underlying set* of $\mathcal{A}$;

(ii) a set $R^{\mathcal{A}} \subseteq A^{n_R}$ for each $R \in \mathcal{R}$,

(iii) a function $f^{\mathcal{A}} : A^{n_f} \longrightarrow A$ for each $f \in \mathcal{F}$;

(iv) an element $c^{\mathcal{A}} \in A$ for each $c \in \mathcal{C}$.

We refer to $R^{\mathcal{A}}$, $f^{\mathcal{A}}$ and $c^{\mathcal{A}}$ as the *interpretations* of the symbols R, f and c.

Example. $L_g = \{\cdot, e\}$, where $\cdot$ is a binary function symbol and e is a constant symbol. An L_g-structure $\mathcal{G} = (G, \cdot^{\mathcal{G}}, e^{\mathcal{G}})$ will be a set G equipped with a binary function $\cdot^{\mathcal{G}}$ and a constant $e^{\mathcal{G}}$. For example, $\mathcal{G} = (\mathbb{R}, \cdot, 1)$ is an L_g-structure where we interpret $\cdot^{\mathcal{G}}$ as multiplication and $e^{\mathcal{G}}$ as 1. L_g is the language suitable for groups, but our example shows that not every L_g-structure is a group.

Remark. By a many-sorted language L, we mean a set $\mathcal{S}$ of *sorts*, a set $\mathcal{R}$ of sorted relation symbols, a set $\mathcal{F}$ of sorted function symbols, and a set $\mathcal{C}$ of sorted constant symbols. What this means is that each $R \in \mathcal{R}$ comes together with a sequence $(S_1, ..., S_n)$ of sorts where $n \geq 1$, each $f \in \mathcal{F}$ comes together with a sequence $(S_1, ..., S_n, S)$ of sorts where $n \geq 1$, and each $c \in \mathcal{C}$ comes together with a sort S.

By an L-structure $\mathcal{A}$ we mean a family $(S^{\mathcal{A}} : S \in \mathcal{S})$ of nonempty sets, together with, for each $R \in \mathcal{R}$ of sort $(S_1, ..., S_n)$ a set $R^{\mathcal{A}} \subseteq S_1^{\mathcal{A}} \times ... \times S_n^{\mathcal{A}}$, for each $f \in \mathcal{F}$ of sort $(S_1, ..., S_n, S)$ a function $f^{\mathcal{A}} : S_1^{\mathcal{A}} \times ... \times S_n^{\mathcal{A}} \to S^{\mathcal{A}}$, and for each $c \in \mathcal{C}$ of sort S an element $c^{\mathcal{A}} \in S^{\mathcal{A}}$.

An one-sorted language is a many-sorted language with only one sort, that is $\mathcal{S}$ is a singleton, say $\{S\}$. In that case the sort of a relation symbol is just a natural number $n \geq 1$ and likewise for the sort of a function symbol. We confine ourselves to one-sorted languages in this text, but all results hold true for many-sorted languages.

Example. By an *incidence spatial geometry* we mean a three-sorted structure $\mathcal{A} = (P, L, \Pi; R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, R_3^{\mathcal{A}})$ where P, L and Π are non-empty sets whose elements are called *points*, *lines* and *planes*, respectively. In addition, $R_1^{\mathcal{A}}$ is a intersort binary relation between points and lines ($R_1^{\mathcal{A}} \subset P \times L$), $R_2^{\mathcal{A}}$ is a intersort binary relation between points and planes ($R_2^{\mathcal{A}} \subset P \times \Pi$) and $R_3^{\mathcal{A}}$ is a intersort binary relation between lines and planes ($R_3^{\mathcal{A}} \subset L \times \Pi$). Those three relations are incidence relations.

In the next section, we define variables. In the case of many sorted languages, variables for differente sorts are written using different letter sets.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures with underlying sets A and B, respectively. An *L-embedding* $\eta : \mathcal{A} \longrightarrow \mathcal{B}$ is an one-to-one map $\eta : A \longrightarrow B$ that preserves and reflects the interpretation of all the symbols of L. We mean,

i) $\eta(f^{\mathcal{A}}(a_1, ..., a_{n_f})) = f^{\mathcal{B}}(\eta(a_1), ..., \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, ..., a_{n_f} \in A$.

ii) $(a_1, ..., a_{n_R}) \in R^{\mathcal{A}}$ if and only if $(\eta(a_1), ..., \eta(a_{n_R})) \in R^{\mathcal{B}}$ for all $R \in \mathcal{R}$ and $a_1, ..., a_{n_R} \in A$.

iii) $\eta(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for all $c \in \mathcal{C}$.

A bijective L-embedding is called an *L-isomorphism*. If $A \subseteq B$ and the inclusion map is an L-embedding, we say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$ or that $\mathcal{B}$ is an *extension* of $\mathcal{A}$ ($\mathcal{A} \subseteq \mathcal{B}$).

Examples.

(1) $(\mathbb{Z}; +; 0) \subseteq (\mathbb{R}; +; 0)$

(2) if $\eta : \mathbb{Z} \longrightarrow \mathbb{R}$ is the function $\eta(x) = e^x$, then η is an L_g-embedding of $(\mathbb{Z}; +; 0)$ into $(\mathbb{R}; \times; 1)$.

2.2 FORMULAS AND SENTENCES

It is of interest to know properties of structures. For that, we use the language L to create terms and formulas that will describe properties of L-structures.

We will also use the following logical symbols:

- the connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$.
- the quantifiers $\forall$ and $\exists$.
- an infinite collection of variables indexed by the natural numbers $\mathbb{N}$ ($v_1, v_2, ...$).
- symbols to indicate grouping: parentheses.

Terms and formulas will be strings of symbols using the symbols of L and the logical symbols above.

Definition. An L-*term* is defined as follows:

(1) each variable is an L-term;

(2) a constant symbol is an L-term;

(3) if $t_1, ..., t_{n_f}$ are L-terms and $f \in \mathcal{F}$, then $f(t_1, ..., t_{n_f})$ is an L-term;

(4) a string of symbols is an L-term if it can be shown to be an L-term by a finite number of applications of (1); (2) and (3).

Definition. Let $\mathcal{A}$ be an L-structure and t be an L-term and $\vec{v} = (v_1, ..., v_m)$ be a vector of variables. Then we associate to the ordered pair $(t, \vec{v})$ a function $t^{\mathcal{A}} : A^m \to A$ as follows:

(i) if t is the constant symbol c then $t^{\mathcal{A}}(a) = c^{\mathcal{A}}$ for $a \in A^m$.

(ii) If t is the variable v_i then $t^{\mathcal{A}}(a) = a_i$ for $a = (a_1, ..., a_m) \in A^m$.

(iii) If $t = f(t_1, ..., t_{n_f})$ where $f \in \mathcal{F}$ and $t_1, ..., t_{n_f}$ are L-terms, then $t^{\mathcal{A}}(a) = f^{\mathcal{A}}\left(t_1^{\mathcal{A}}(a), ..., t_{n_f}^{\mathcal{A}}(a)\right)$ for $a \in A^m$.

Example. Let $L = \{f, g, c\}$, where f is a unary function symbol, g is a binary function symbol, and c is a constant symbol. We will consider the L-terms $t_1 = g(v_1, c)$, $t_2 = f(g(c, f(v_1)))$, and $t_3 = g(f(g(v_1, v_2)), g(v_1, f(v_2)))$. Let $\mathcal{A}$ be the L-structure $(\mathbb{R}; \exp, +; 1)$; that is, $f^{\mathcal{A}} = \exp$, $g^{\mathcal{A}} = +$, and $c^{\mathcal{A}} = 1$.
Then

$$t_1^{\mathcal{A}}(a_1) = a_1 + 1, \qquad t_2^{\mathcal{A}}(a_1) = e^{1+e^{a_1}}, \quad \text{and} \quad t_3^{\mathcal{A}}(a_1, a_2) = e^{a_1+a_2} + (a_1 + e^{a_2}).$$

Note that if $\mathcal{B}$ is a second L-structure and $\mathcal{A} \subseteq \mathcal{B}$, then $t^{\mathcal{A}}(a) = t^{\mathcal{B}}(a)$ for t as above and $a \in A^m$.

A term is said to be *variable-free* if no variables occur in it. Let t be a variable-free L-term and $\mathcal{A}$ an L-structure. Then the above gives a nullary function $t^{\mathcal{A}} : A^0 \to A$, identified as usual with its value at the unique element of A^0, so $t^{\mathcal{A}} \in A$. In other words, if t is a constant symbol c, then $t^{\mathcal{A}} = c^{\mathcal{A}} \in A$, where $c^{\mathcal{A}}$ is as in the previous section, and if $t = f(t_1...t_n)$ with n-ary $f \in \mathcal{F}$ and variable-free L-terms $t_1, ..., t_n$, then $t^{\mathcal{A}} = f^{\mathcal{A}}(t_1^{\mathcal{A}}, ..., t_n^{\mathcal{A}})$.

Definition. An *L-formula* is defined as follows:

(1) If t_1 and t_2 are L-terms, then $t_1 = t_2$ is an L-formula.

(2) If R is an n_R-ary relation symbol and t_1, ... ,t_{n_R} are L-terms, then $R(t_1, ..., t_{n_R})$ is a formula. (1) and (2) are called *atomic L-formulas*.

(3) If φ is a L-formula, then $\neg\varphi$ is a L-formula.

(4) If φ and ψ are L-formulas then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$.

(5) If v_i is a variable and φ is a formula, then $(\exists v_i)\varphi$ and $(\forall v_i)\varphi$ are formulas.

(6) A string of symbols is a formula if it can be shown to be a formula by a finite numbers of applications of (1), (2), (3), (4) and (5).

We say that an instance of a variable v_i in a formula φ is *free* if it is not inside the scope of a $\exists v_i$ or $\forall v_i$ quantifier; otherwise, we say it is *bound*. By scope of the quantifier (Qv_i) we mean the subformula φ in $Qv_i\varphi$ (item 5 above).

So, it is said that v_i is a *free variable* in φ if it appears free in φ.

Example. All instances of x in the formula $\forall x(x = y \vee \exists y(x \neq y))$ are bound, while the first instance of y is free and the other two are bound. Hence y is the only free variable of this formula.

Definition. We call a formula a *sentence* if it has no free variables.

From now on, $\varphi(v_1, v_2, ..., v_n)$ will indicate a formula φ such that all variables that occur free in φ are among $v_1, ..., v_n$.

At this point, it is necessary to enlarge our L-structure $\mathcal{A}$ by introducing names. For this, let $C \subseteq A$. We extend L to a language $L(C)$ by adding a constant symbol $\underline{c}$ for each

$c \in C$. As a consequence, $\mathcal{A}$ is made an $L(C)$-structure by keeping the same underlying set and interpretations of symbols of L, and by interpreting each name $\underline{c}$ as the element $c \in C$.

Definition. We can now define what it means for an $L(A)$-sentence σ to be *true in the L-structure* $\mathcal{A}$ (notation: $\mathcal{A} \models \sigma$, also read as $\mathcal{A}$ *satisfies* σ or σ *holds in* $\mathcal{A}$, or σ *is valid in* $\mathcal{A}$). First we consider atomic $L(A)$-sentences:

(i) $\mathcal{A} \models t_1 = t_2$ if and only if $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$, for variable-free $L(A)$-terms t_1, t_2 that is, t_1 and t_2 are built of constant and function symbols only.

(ii) $\mathcal{A} \models R(t_1, ..., t_{n_R})$ if and only if $\left(t_1^{\mathcal{A}}, ..., t_{n_R}^{\mathcal{A}}\right) \in R^{\mathcal{A}}$, for $R \in \mathcal{R}$, and variable-free $L(A)$-terms $t_1, ..., t_{n_R}$;

We extend the definition inductively to arbitrary $L(A)$-sentence as follows:

(i) Suppose $\sigma = \neg\sigma_1$. Then $\mathcal{A} \models \sigma$ if and only if $\mathcal{A} \not\models \sigma_1$.

(ii) Suppose $\sigma = \sigma_1 \vee \sigma_2$. Then $\mathcal{A} \models \sigma$ if and only if $\mathcal{A} \models \sigma_1$ or $\mathcal{A} \models \sigma_2$.

(iii) Suppose $\sigma = \sigma_1 \wedge \sigma_2$. Then $\mathcal{A} \models \sigma$ if and only if $\mathcal{A} \models \sigma_1$ and $\mathcal{A} \models \sigma_2$.

(iv) Suppose $\sigma = \exists v \varphi(v)$. Then $\mathcal{A} \models \sigma$ if and only if $\mathcal{A} \models \varphi(\underline{a})$ for some $a \in A$.

(v) Suppose $\sigma = \forall v \varphi(v)$. Then $\mathcal{A} \models \sigma$ if and only if $\mathcal{A} \models \varphi(\underline{a})$ for all $a \in A$.

Remark. The reader should notice that even if σ is an L-sentence of the form $\exists v \varphi(v)$ or $\forall v \varphi(v)$, the inductive definition above forces us to consider $L(A)$-sentences $\sigma(\underline{a})$. For that reason, we introduced names.

We defined what it means for a *sentence* σ to hold in a given structure $\mathcal{A}$. We now extend this to arbitrary formulas.

First define an $\mathcal{A}$-*instance of a formula* $\varphi = \varphi(v_1, ..., v_m)$ to be an $L(A)$-sentence of the form $\varphi(\underline{a}_1, ..., \underline{a}_m)$ with $a_1, ..., a_m \in A$.

Definition. A formula φ is said to be *valid in* $\mathcal{A}$ (notation: $\mathcal{A} \models \varphi$) if all its $\mathcal{A}$-instances are true in $\mathcal{A}$.

Note that if $\varphi = \varphi(v_1, ..., v_m)$, then

$$\mathcal{A} \models \varphi \Longleftrightarrow \mathcal{A} \models \forall v_1 ... \forall v_m \varphi.$$

2.3 MODELS

In this section L is a language, $\mathcal{A}$ is an L-structure (with underlying set A), and, unless indicated otherwise, t is an L-term, φ, ψ, and θ are L-formulas, σ is an L-sentence, and Σ is a set of L-sentences. We drop the prefix L in "L-term" and "L-formula" and so on, unless this causes confusion.

Definition. We say that $\mathcal{A}$ is a *model of* Σ or Σ *holds in* $\mathcal{A}$ (denoted $\mathcal{A} \models \Sigma$) if $\mathcal{A} \models \sigma$ for each $\sigma \in \Sigma$.

Definition. A *theory* is a set of sentences, which we call "*axioms*".

Example. Vector spaces over a fixed field $\mathcal{K}$ are regarded as structures of the language $L_{\mathcal{K}} = \{0, +, -\} \cup \{f_k : k \in \mathcal{K}\}$, where 0 is a constant symbol, $-$ and $+$ are unary and binary function symbols, respectively; and f_k are unary function symbols which represent multiplication by the scalars $k \in \mathcal{K}$. Let's denote this language by $L_{\mathcal{K}}$.

Consider the following sets of $L_{\mathcal{K}}$-sentences:

(1) $\forall x \forall y \forall z((x + y) + z = x + (y + z))$

(2) $\forall x \forall y(x + y = y + x)$

(3) $\forall x(x + 0 = x)$

(4) $\forall x(x + (-x) = 0)$

(5) $\forall x(f_0(x) = 0 \wedge f_1(x) = x)$

(6) $\forall x \forall y(f_k(x + y) = f_k(x) + f_k(y))$, one sentence for each $k \in \mathcal{K}$

(7) $\forall x(f_{k_1+k_2}(x) = f_{k_1}(x) + f_{k_2}(x))$, one sentence for each $k_1, k_2 \in \mathcal{K}$

(8) $\forall x(f_{k_1}(f_{k_2}(x)) = f_{k_1 \cdot k_2}(x))$, one sentence for each $k_1, k_2 \in \mathcal{K}$

These axiomatize the class of *vector spaces over* $\mathcal{K}$. We denote the $L_{\mathcal{K}}$-theory of all such spaces by $T_{\mathcal{K}}$.

2.4 ULTRAPRODUCTS

In this section we describe the construction that will occupy our attention for the most of the rest of this work, namely the ultraproduct.

Before talking about ultraproducts, we must define what an ultrafilter is.

Definition. Let I be a non-empty set and let $\mathcal{P}(I)$ denote the power set of I. A $filter$ on I is a collection $F \subset \mathcal{P}(I)$ such that:

(i) $I \in F, \emptyset \notin F$;

(ii) if $A, B \in F$, then $A \cap B \in F$;

(iii) if $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

Intuitively a filter is a collection of "big" subsets of I.
We say that a filter U on I is an $ultrafilter$ if

(iv) $X \in U$ or $I \setminus X \in U$ for all $X \subseteq I$.

Definition. Let I be a nonempty set. A subset F of $\mathcal{P}(I)$ is said to have the $finite$ $intersection\ property$ (FIP) if $F \neq \emptyset$ and no intersection of finitely many members of F is empty.

Remark. F has the FIP if and only if the set F' of all finite intersections of sets from F satisfies $\emptyset \notin F'$.

Lemma 2.1. *Every $S \subseteq \mathcal{P}(I)$ with the FIP can be extended to an ultrafilter.*

Proof. See ([15], p. 45). This is an example of using Zorn's Lemma.

□

Definition. An ultrafilter U is a $free\ ultrafilter$ if no finite sets belong to U.

Note the lemma above implies the existence of free ultrafilters on infinite I by extending $\{I \setminus \{x\} : x \in I\}$ to an ultrafilter.

Definition. Let L be a language and suppose that I is an infinite set. Suppose that $\mathcal{A}_i$ is an L-structure for each $i \in I$. Let U be an ultrafilter on I. We define a new structure $\mathcal{A}/U = \left(\prod \mathcal{A}_i\right)/U$, which we call the *ultraproduct* of the $\mathcal{A}_i$ using U.
First, define a relation U on

$$\prod_{i \in I} A_i = \{a : I \to \bigcup A_i : a(i) \in A_i \text{ for all } i\}$$

by $a \sim_U b$ if only if $\{i \in I : a(i) = b(i)\} \in U$. We see that $\sim_U$ is an equivalence relation.

Let us show transitivity (reflexivity and symmetry are easier). Since $a \sim_U b$ and $b \sim_U c$, we have $S = \{i \in I : a(i) = b(i)\} \in U$ and $T = \{i \in I : b(i) = c(i)\} \in U$. Considering that $S \cap T \subseteq \{i \in I : a(i) = c(i)\} \subseteq I$, and that $S, T \in U$, and that U is closed under intersections and supersets we get $\{i \in I : a(i) = c(i)\} \in U$. Thus $a \sim_U c$.

The universe of $\mathcal{A}/U$ will be $\left(\prod_{i \in I} A_i\right)/U$, the collection of all $\sim_U$ equivalence classes.

Remark. We write $\bar{a}/U$ for $(a_1/U, \dots, a_n/U)$.

Now, given a constant symbol c of L, set $c^{\mathcal{A}/U} = \left(c^{\mathcal{A}_i} : i \in I\right)/U$.

Given an n-ary function symbol f of L and $\bar{a} = (a_1, \dots, a_n)$ in $\left(\prod A_i\right)^n$, set $f^{\mathcal{A}/U}(\bar{a}/U) = (f^{\mathcal{A}_i}(a_1(i), \dots, a_n(i)) : i \in I)/U$, i.e., $f^{\mathcal{A}/U}(\bar{a}/U) = (f^{\mathcal{A}_i}(\bar{a}))/U$. Let us show this is well-defined. Take $a_1, \dots, a_n, b_1, \dots, b_n \in \Pi A_i$ and $a_i \sim_U b_i$ for $i = 1, \dots, n$. Define $a_{n+1}(i) = f^{\mathcal{A}_i}(a_1(i), \dots, a_n(i))$ and $b_{n+1}(i) = f^{\mathcal{A}_i}(b_1(i), \dots, b_n(i))$ for $i \in I$. Then $a_{n+1} \sim_U b_{n+1}$ follows because we always have

$$\{i \in I : a_1(i) = b_1(i), \dots, a_n(i) = b_n(i)\} \subseteq \{i \in I : a_{n+1}(i) = b_{n+1}(i)\},$$

and so the set on the right is in U by closure under intersections and supersets.

Given an n-ary relation symbol R of L and $\bar{a} = (a_1, \dots, a_n)$ in $\left(\prod A_i\right)^n$, set $\bar{a}/U \in R^{\mathcal{A}/U} \Leftrightarrow \{i \in I : \bar{a}(i) \in R^{\mathcal{A}_i}\} \in U$. Let us show that this is well-defined by checking this does not depend on the choice of representative elements, i.e., $\{i \in I : (a_1(i), \dots, a_n(i)) \in R^{\mathcal{A}_i}\}$ iff $\{i \in I : (b_1(i), \dots, b_n(i)) \in R^{\mathcal{A}_i}\}$. For that, take $a_1, \dots, a_n, b_1, \dots, b_n$ as before. Suppose $\{i \in I : (a_1(i), \dots, a_n(i)) \in R^{\mathcal{A}_i}\} \in U$. Now, we have

$$\bigcap_{k=1}^{n} \{i \in I : a_k(i) = b_k(i)\} \cap \{i \in I : (a_1(i), \dots, a_n(i)) \in R^{\mathcal{A}_i}\} \subseteq \{i \in I : (b_1(i), \dots, b_n(i)) \in R^{\mathcal{A}_i}\}$$

Since all of the sets on the left are in U, again by the filter axioms, we get the set on the right in U as well. Thus, $\{i \in I : (a_1(i), ..., a_n(i)) \in R^{\mathcal{A}_i}\} \in U \Rightarrow \{i \in I : (b_1(i), ..., b_n(i)) \in R^{\mathcal{A}_i}\} \in U$. Analogously, we get $\{i \in I : (b_1(i), ..., b_n(i)) \in R^{\mathcal{A}_i}\} \in U \Rightarrow \{i \in I : (a_1(i), ..., a_n(i)) \in R^{\mathcal{A}_i}\} \in U$.

Now, for ultraproducts, a term t involving the variables $v_1, \cdots, v_n$ is interpreted as a function $t^{\mathcal{A}/U} : (\mathcal{A}/U)^n \to \mathcal{A}/U$. Then based on all facts exposed above we can state

Lemma 2.2. . *For $a_1/U, \cdots, a_n/U$ in $\mathcal{A}/U$,*

$$t^{\mathcal{A}/U}(a_1/U, \cdots, a_n/U) = \left(t^{\mathcal{A}_i}(a_1(i), \cdots, a_n(i)) : i \in I\right)/U.$$

Proof. Let us show its is well defined.
Take $a_1, \cdots, a_n, b_1, \cdots, b_n \in \prod A_i$ and $a_j \sim_U b_j$ for $j = 1, \cdots, n$.
For $t = c$.
Define $c'(i) = t^{\mathcal{A}_i}(a_1(i), \cdots, a_n(i))$ and $c''(i) = t^{\mathcal{A}_i}(b_1(i), \cdots, b_n(i))$ for $i \in I$. Then $c' \sim_U c''$ follows from

$$\{i \in I : a_1(i) = b_1(i), ..., a_n(i) = b_n(i)\} \subseteq \{i \in I : c'(i) = c''(i)\} \in U.$$

For $t = v_k$ for some $k \in \{1; \cdots; n\}$.
Define $a_k(i) = t^{\mathcal{A}_i}(a_1(i), \cdots, a_n(i))$ and $b_k(i) = t^{\mathcal{A}_i}(b_1(i), \cdots, b_n(i))$ for $i \in I$. Since $a_j \sim_U b_j$ for $j = 1, \cdots, n$ then $a_k \sim_U b_k$.

For $t = f(t_1, \cdots, t_m)$ and assuming the statements hold for $t_1, \cdots, t_m$. We write $a(i)$ for $(a_1(i), \cdots, a_n(i))$.
Define $a_{m+1}(i) = f^{\mathcal{A}_i}\left(t_1^{\mathcal{A}_i}(a(i)), \cdots, t_m^{\mathcal{A}_i}(a(i))\right)$ and $b_{m+1}(i) = f^{\mathcal{A}_i}\left(t_1^{\mathcal{A}_i}(b(i)), \cdots, t_m^{\mathcal{A}_i}(b(i))\right)$ for $i \in I$. Then $a_{m+1} \sim_U b_{m+1}$ follows from

$$\{i \in I : t_1^{\mathcal{A}_i}(a(i)) = t_1^{\mathcal{A}_i}(b(i)), \cdots, t_m^{\mathcal{A}_i}(a(i)) = t_m^{\mathcal{A}_i}(b(i))\} \subseteq \{i \in I : a_{m+1}(i) = b_{m+1}(i)\} \in U.$$

□

The following theorem is due to *Jerzy Łoś*. He proved that any first-order formula is true in the ultraproduct $\mathcal{A}/U$ if and only if the set of indices i such that the formula is true in $\mathcal{A}_i$ is a member of U. More precisely:

Theorem 2.3. *(Łoś's Theorem) Let $\varphi(v_1, ..., v_n)$ be an L-formula and $\bar{a} = (a_1, ..., a_n)$ in $\left(\prod A_i\right)^n$. Then $\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U)$ if and only if $\{i \in I : \mathcal{A}_i \models \varphi(a_1(i), ..., a_n(i))\} \in U$.*

Proof. The proof is by induction on the number of logical symbols in φ, and we use $\neg, \wedge, \exists$ as a complete set of connectives and quantifiers [1].
Consider first t_1 and t_2 be two terms whose free variables are among $v_1, ...v_n$.
Let $\varphi \equiv t_1 = t_2$. We have

$$\begin{aligned}
&\mathcal{A}/U \models (t_1 = t_2)(a_1/U, ..., a_n/U) \\
\Leftrightarrow\ & t_1^{\mathcal{A}/U}(a_1/U, ..., a_n/U) = t_2^{\mathcal{A}/U}(a_1/U, ..., a_n/U) \\
\Leftrightarrow\ & \{i \in I : t_1^{\mathcal{A}_i}(a_1(i), ..., a_n(i)) = t_2^{\mathcal{A}_i}(a_1(i), ..., a_n(i))\} \in U \\
\Leftrightarrow\ & \{i \in I : \mathcal{A}_i \models (t_1 = t_2)(a_1(i), ..., a_n(i))\} \in U.
\end{aligned}$$

Next consider an L-atomic formula $\varphi \equiv R(t_1, ..., t_m)$ where R is an m-ary relation symbol and the free variables of $t_1, ..., t_m$ are among $v_1, ..., v_n$. We have

$$\begin{aligned}
&\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \\
\Leftrightarrow\ & R^{\mathcal{A}/U}(t_1^{\mathcal{A}/U}(a_1/U, ..., a_n/U), ..., t_m^{\mathcal{A}/U}(a_1/U, ..., a_n/U)) \\
\Leftrightarrow\ & \{i \in I : R^{\mathcal{A}_i}(t_1^{\mathcal{A}_i}(a_1(i), ..., a_n(i))..., t_m^{\mathcal{A}_i}(a_1(i), ..., a_n(i)))\} \in U \\
\Leftrightarrow\ & \{i \in I : \mathcal{A}_i \models \varphi(a_1(i), ..., a_n(i))\} \in U.
\end{aligned}$$

Now, suppose that the theorem holds for φ_1 and φ_2.
If $\varphi \equiv \neg\varphi_1$, we have

$$\begin{aligned}
&\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \Leftrightarrow \mathcal{A}/U \models \neg\varphi_1(a_1/U, ..., a_n/U) \\
\Leftrightarrow\ & \{i \in I : \mathcal{A}_i \models \varphi_1(a_1(i), ..., a_n(i))\} \notin U
\end{aligned}$$

Then,

$$\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \Leftrightarrow I \setminus \{i \in I : \mathcal{A}_i \models \varphi_1(a_1(i), ..., a_n(i))\} \in U$$

Thus,

$$\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \Leftrightarrow \{i \in I : \mathcal{A}_i \models \varphi(a_1(i), ..., a_n(i))\} \in U.$$

If $\varphi \equiv \varphi_1 \wedge \varphi_2$, we have

$$\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \Leftrightarrow \mathcal{A}/U \models \varphi_1(a_1/U, ..., a_n/U) \text{ and } \mathcal{A}/U \models \varphi_2(a_1/U, ..., a_n/U)$$

If and only if,

$$\{i \in I : \mathcal{A}_i \models \varphi_1(a_1(i), ..., a_n(i))\} \in U \text{ and } \{i \in I : \mathcal{A}_i \models \varphi_2(a_1(i), ..., a_n(i))\} \in U$$

Thus,

1 By equivalence, we have: $\forall x\varphi \Leftrightarrow \neg\exists x(\neg\varphi)$, $\varphi \vee \psi \Leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi \Leftrightarrow \neg(\varphi \wedge \neg\psi)$ and $\varphi \leftrightarrow \psi \Leftrightarrow \neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)$

$\{i \in I : \mathcal{A}_i \models \varphi_1(a_1(i), ..., a_n(i))$ and $\mathcal{A}_i \models \varphi_2(a_1(i), ..., a_n(i))\} \in U$ (see below)
It can be written as
$\{i \in I : \mathcal{A}_i \models (\varphi_1 \wedge \varphi_2)(a_1(i), ..., a_n(i))\} \in U$.

Remark. If the two sets belong to U, then their intersection belongs to U. Conversely, if $X \cap Y \in U$, note $X, Y \supseteq X \cap Y$, so $X, Y \in U$.

Finally, suppose $\varphi \equiv \exists v \varphi_1(v, v_1, ..., v_n)$.

First, we note that $\mathcal{A}/U \models \varphi(a_1/U, ..., a_n/U) \Leftrightarrow \exists b/U \in \Pi A_i/U$ such that $\{i \in I : \mathcal{A}_i \models \varphi_1(b(i), a_1(i), ..., a_n(i))\} \in U$.

We want to show this last condition is equivalent to $\{i \in I : \mathcal{A}_i \models \exists v \varphi_1(v, a_1(i), ..., a_n(i))\} \in U$.

Let us call the above sets S_1 and S_2, respectively. The existence of b such that $S_1 \in U$ implies $S_2 \in U$ because of $S_1 \subset S_2$ and U is closed under enlargements.

Now, for each $i \in S_2$ choose some element $b(i) \in A_i$ that satisfies $\mathcal{A}_i \models \varphi_1(b(i), a_1(i), ..., a_n(i))$. For every $j \in I \setminus S_2$, let $b(j)$ be arbitrary in A_j.
By the choice of b, we have proved that if $S_2 \in U$ then $S_1 \in U$.

□

Corollary 2.4. *(The Compactness Theorem) A set Σ of L-sentences has a model if and only if each finite subset of Σ has a model.*

Proof. ($\Rightarrow$) If Σ has a model, then each subset has the same model.

($\Leftarrow$) Let I be the set of all non-empty finite subsets of Σ, we mean $I = \{i : i \subset \Sigma, i \neq \phi$ and i is finite $\}$. By hypothesis, there is a non-empty L-structure $\mathcal{A}_i$ for each $i \in I$ such that $\mathcal{A}_i \models i$.
For each set $i \in I$ let $i^* = \{i' \in I : i \subseteq i'\}$. Each collection i^* is non-empty. Also, for any finite subset, say $\{i_1, ..., i_n\} \subset I$, we have $i_1 \cup ... \cup i_n \in i_1^* \cap ... \cap i_n^*$ and hence the collection $\{i^* : i \in I\}$ has the FIP (finite intersection property).
Therefore, this collection can be extended to an ultrafilter U on I.
We show that the ultraproduct $\prod_{i \in I} \mathcal{A}_i/U$ is a model of Σ.

Suppose $\sigma \in \Sigma$, then $\{\sigma\} \in I$, say $\{\sigma\} = i_0$. $\mathcal{A}_{i_0} \models \sigma$ and clearly, $i' \models \sigma$ [2], if $i_0 \subseteq i'$.
Hence $i_0^* = \{i \in I : \sigma \in i\} \subseteq \{i \in I : \mathcal{A}_i \models \sigma\}$.
By the choice of U, we have $i_0^* \in U$ and hence $\{i \in I : \mathcal{A}_i \models \sigma\} \in U$.
Therefore, by Łoś's theorem $\prod_{i \in I} \mathcal{A}_i/U \models \sigma$. Since σ was arbitrary in Σ, we have $\prod_{i \in I} \mathcal{A}_i/U \models \Sigma$.

□

Example. Let $\mathcal{A} = (\mathbb{R}; +, \cdot, -; <; 0)$ be the structure for the real numbers and U be a free ultrafilter on $\mathbb{N}$. Let $\mathcal{A}/U = (\mathbb{R}^*; +, \cdot, -; <; 0)$ be the ultrapower of $\mathcal{A}$. Remember that $\mathbb{R}^* = \prod \mathbb{R}/U = \{(\bar{x})_U : \bar{x} \in \prod \mathbb{R}\}$. Since $\mathbb{R}$ is a ordered field, so is $\mathbb{R}^*$. Note that we can embed $\mathbb{R}$ in $\mathbb{R}^*$ by taking $x \mapsto E(x) = (\bar{x})_U$, where $\bar{x} = (x, x, ...)$.

Take $\alpha = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right)$. We show that α_U is a positive infinitesimal. Note that $\{n \in \mathbb{N} : \alpha_n > 0\} = \{n \in \mathbb{N} : \frac{1}{n} > 0\} = \mathbb{N} \in U$, so by Łoś we have $\alpha_U > E(0)$.
For $\epsilon > 0$ in $\mathbb{R}$, $\{n \in \mathbb{N} : \alpha_n < \epsilon\} = \{n \in \mathbb{N} : \frac{1}{n} < \epsilon\}$ is cofinite, hence it belongs to U and by Łoś we have $\alpha_U < E(\epsilon)$

Therefore $0_{\mathbb{R}^*} < \alpha_U < \epsilon_{\mathbb{R}^*}$ for every real $\epsilon > 0$.

2 We say σ is a *logical consequence* of Σ ($\Sigma \models \sigma$) if $\mathcal{A} \models \sigma$ for all models $\mathcal{A}$ of Σ

3 CONTINUOUS LOGIC

In this chapter, we develop continuous logic, a variant of the logic described before. The mostly of definitions and examples are taken from [2] because we consider this text contains all the necessary to understand the model-theoretic properties of the metric structures. The basic idea is doing it parallel to the usual classical logic, once one enlarges the set of possible truth values from $\{0,1\}$ to $[0,1]$. Since $x = y \Leftrightarrow d(x,y) = 0$, then we assign 0 to "*true*" and 1 to "*false*", while positive values between them represent distinct degrees of falsity, errors or approximations.

From now on, we take metric spaces as our underlying sets to define *metric prestructures*. By taking their completion whenever necessary, we focus always on *metric structures* whose underlying domains are complete metric spaces.

3.1 METRIC STRUCTURES

Definition. Let M be a metric space. By a *relation on* M, we understand a uniformly continuous function from M^n into $[0,1]$. By a *function on* M^n, we understand a uniformly continuous function from M^n into M (in both cases, $n \in \mathbb{N}$ plays the role of arity).

Definition. A *metric structure* $\mathcal{M}$ based on (M,d), which is a complete, 1-bounded metric space, consists of a family $(R_i : i \in I)$ of relations on M, a family $(F_j : j \in J)$ of functions on M and a family $(a_k : k \in K)$ of distinguished elements of M. We will often denote it as

$$\mathcal{M} = (M, d, R_i, F_j, a_k : i \in I, j \in J, k \in K)$$

Our theory also applies to *many sorted* metric structures and we proceed in the same way as classical logic. However, in this chapter they will just appear as examples.

Examples. The most of them taken from [2] which we quote verbatim. We have:

(1) A complete, 1-bounded metric space (M, d) with no additional structure.

(2) A structure $\mathcal{M}$ in the usual sense from classical logic. One puts the discrete metric on the underlying set ($d(a, b) = 1$ when a, b are distinct) and a relation is considered as a predicate taking values in the set $\{0, 1\}$.

(3) Let $(X, \|\cdot\|)$ be a normed space. We can regard it as a metric prestructure in the following way. The sorts are $B_n := \{x \in X : \|x\| \leq n\}$, indexed by $n \in \mathbb{N}$. It comes equipped with distinguished elements $0^n \in B_n$ ($0^n = 0 \in X$, $\forall n \in \mathbb{N}$), functions $+_n : B_n \times B_n \to B_{2n}$ (the restrictions of vector space addition to bounded balls with center at the origin), $-_n : B_n \to B_n$ (the restrictions of the additive inverse unary operation to balls with center at the origin), and the inclusion mappings $I_{mn} : B_m \to B_n$, where $m < n$ (they are functions that tie together the different sorts). The metrics can be thought of as distinguished predicates, and must be rescaled to 1.

(4) The unit ball B of a Banach space X over $\mathbb{R}$ or $\mathbb{C}$: we can work with $\hat{+} : B \times B \to B$ defined as $\hat{+}(x, y) = (x + y)/2$. Also, we may take the maps $f_\alpha : B \to B$ where $f_\alpha(x) = \alpha x$ for $|\alpha| \leq 1$; the norm may be included as a predicate, and we may include the additive identity 0_X as a distinguished element. Note that since $\|x \hat{+} y\| = \frac{1}{2}\|x + y\| \leq \frac{1}{2}(\|x\| + \|y\|) \leq \frac{1}{2}(1 + 1) = 1$, we ensure that $x \hat{+} y \in B$. Similarly, $f_\alpha(x) \in B$, for all $x \in B$ and each scalar α such that $|\alpha| \leq 1$. (One could also consider $f_{\alpha,\beta}(x, y) = \alpha x + \beta y$ as maps where each pair of scalars satisfied $|\alpha| + |\beta| \leq 1$).

(5) If $(\Omega, \mathcal{B}, \mu)$ is a probability space, we may construct a metric structure $\mathcal{M}$ from it, based on the metric space (M, d) in which M is the measure algebra of $(\Omega, \mathcal{B}, \mu)$ (elements of $\mathcal{B}$ modulo set of measure 0) and d is defined to be the measure of the symmetric difference. As a predicate on M we take the measure μ, and as distinguished elements the 0 and 1 of M. We include the boolean operation as functions in the structure.

Now, having a metric structure $\mathcal{M}$, we associate to it a *language* L as follows:

- To each relation R of $\mathcal{M}$ we associate a *predicate symbol* P and an integer $n_P \in \mathbb{N}$ (the arity of R); we denote R by $P^{\mathcal{M}}$.

- To each function F of $\mathcal{M}$ we associate a *function symbol* f and an integer $n_f \in \mathbb{N}$ (the arity of F); we denote F by $f^{\mathcal{M}}$.

- To each distinguished element a of $\mathcal{M}$ we associate a *constant symbol* c; we denote a by $c^{\mathcal{M}}$.

Moreover, the language must give more information, i.e., for each predicate (function, respectively), it must provide a *modulus of uniform continuity* Δ_P (Δ_f, respectively). Also, we will denote the metric d given by $\mathcal{M}$ as $d^{\mathcal{M}}$.

Remark. A modulus of uniform continuity is a specified way of providing a δ for a given ϵ in the definition of uniformly continuous functions, say $f : (M, d_1) \to (N, d_2)$:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \forall y \; (d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) \leq \epsilon).$$

Formally, it is a function $\Delta : (0, 1] \to (0, 1]$ such that

$$\forall \epsilon > 0 \; \forall x \forall y \; (d_1(x, y) < \Delta(\epsilon) \Rightarrow d_2(f(x), f(y)) \leq \epsilon).$$

The reason for that choice of strict and non-strict inequalities is just operational, and will become clear in the proof that an ultraproduct of functions is again a function (page 25).

After these requirements are all met and when the predicate symbol (also function and constant symbols) correspond exactly to the relation (function and distinguished elements, respectively) of which $\mathcal{M}$ consists, and their moduli of uniform continuity match those of $\mathcal{M}$, we say that $\mathcal{M}$ is an *L-structure*.

Examples. Here we quote two examples taken from [2].

(1) Let $L^p(X, U, \mu)$ be the space of (equivalence classes of) U-measurable functions $f : X \to \mathbb{R}$ such that $\|f\| = \left(\int |f|^p d\mu\right)^{1/p} < \infty$. We build a metric structure $\mathcal{M} = \left((B_n : n \geq 1), 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \wedge, \vee, \frac{1}{n}\|\cdot\|\}\right)$ whose $B_n = \{f \in L^p(X, U, \mu) : \|f\| \leq n\}$ and $I_{mn} : B_m \to B_n$ is the inclusion map from $m < n$. The

metric on each B_n is given by $d_n(f,g) = \|f-g\|/2n$. The diameter of B_n is 1 and the values of the predicate $\frac{1}{n}\|\cdot\|$ on B_n are in $[0,1]$. The operations $+, -, \wedge, \vee$ map $B_n \times B_n$ into B_{2n}. For $r \in \mathbb{R}$ with $k-1 < |r| \leq k$, where $k \geq 1$ is an integer, the operation λ_r maps B_n into B_{kn}. The moduli of uniform continuity for the norm and for the inclusion maps I_{mn} are all given by $\Delta(\epsilon) = \epsilon$. The moduli of uniform continuity for $+, -, \wedge, \vee$ are all given by $\Delta'(\epsilon) = \epsilon/2$. For $r \in \mathbb{R}$ with $k-1 < |r| \leq k$, where k is an integer ≥ 1, the modulus of uniform continuity of λ_r is given by $\Delta_{\lambda_r}(\epsilon) = \epsilon/k$.

(2) Given a probability space $(\Omega, \mathcal{B}, \mu)$, we build a metric structure (called a *probability structure*) $\mathcal{M} = (\hat{\mathcal{B}}, 0, 1, \cdot^c, \cap, \cup, \mu)$ whose $\hat{\mathcal{B}}$ is the collection of the equivalences classes of $\mathcal{B}$ modulo $\sim_\mu$ and the metric is given by $d([A]_\mu, [B]_\mu) = \mu(A \triangle B)$. Here 0 is the event of measure zero, 1 the event of measure one, and $\cdot^c, \cap, \cup$ are the Boolean operations induced on $\hat{\mathcal{B}}$. The moduli of uniform continuity for $\cdot^c$ is the identity $\Delta(\epsilon) = \epsilon$ and the moduli of uniform continuity for $\cup$ and $\cap$ are given by $\Delta'(\epsilon) = \epsilon/2$.

3.2 CONTINUOUS LOGIC FOR METRIC STRUCTURES

Given a language L, we are able to define *L-terms* and *atomic L-formulas* as usual.

L-terms:

- Variables and constant symbols are L-terms.
- If f is an n-ary function symbol and $t_1, ..., t_n$ are L-terms, then $f(t_1, ..., t_n)$ is an L-term.

Atomic L-formulas:

- $P(t_1, ..., t_n)$, in which P is an n-ary predicate symbol and t_1,...,t_n are L-terms.
- $d(t_1, t_2)$, in which t_1 and t_2 are L-terms.

Remark. We note that the symbol d for the metric is treated as a binary relation symbol in the same way that we did with equality symbol "=" in classical logic because

$d(x, y) = 0 \Leftrightarrow x = y$ and that is why 0 is "*true*".

L-Formulas are constructed from atomic L-formulas by induction as in classical logic. However, since the truth values lie in $[0,1]$, we need to adapt our connectives and quantifiers. So, continuous functions $u : [0,1]^n \to [0,1]$ play the role of connectives and $\sup_x$ and $\inf_x$ act like quantifiers in place of $\forall x$ and $\exists x$, respectively, in the following way: if φ is a formula with free variables $x, \vec{y}$, say, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas with free variables $\vec{y}$. Now given a structure $\mathcal{M}$, any formula φ induces a function $\varphi^{\mathcal{M}}$ into $[0,1]$.

Definition. We define an *L-formula* as follows:

- Atomic L-formulas are L-formulas.
- If $u : [0,1]^n \to [0,1]$ is continuous and $\varphi_1, ..., \varphi_n$ are L-formulas, then $u(\varphi_1, ..., \varphi_n)$ is a L-formula.
- If φ is an L-formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas.

Remark. Since every continuous $f : [0,1] \to [0,1]$ which is 0 on $(0,1]$ also satisfies $f(0) = 0$, we realize there is no proper negation connective in continuous logic and then there is no direct way to express implications between conditions. This is inconvenient in applications, since many natural properties in mathematics are stated using implications. However, we can use the following fact to overcome this issue (see page 52 for definition of ω-saturation):

Proposition 3.1. *Suppose that $\mathcal{M}$ is an ω-saturated L-structure, and $\varphi(\bar{v})$ and $\psi(\bar{v})$ are two L-formulas, where $\bar{v}$ is an n-tuple of variables. Then the following are equivalent:*

1. *For all $\bar{a} \in M^n$, if $\varphi^{\mathcal{M}}(\bar{a}) = 0$, then $\psi^{\mathcal{M}}(\bar{a}) = 0$;*
2. *There is an increasing, continuous function $\alpha : [0,1] \to [0,1]$ satisfying $\alpha(0) = 0$ so that, for all $\bar{a} \in M^n$, we have $\psi^{\mathcal{M}}(\bar{a}) \leq \alpha\left(\varphi^{\mathcal{M}}(\bar{a})\right)$.*

Proof. See [2], p. 40. □

Now, since (2) $\Rightarrow$ (1) is valid always, saturation is necessary to ensure that (1) $\Rightarrow$ (2). Also, this fact tells us that the second condition is indeed expressible by the condition

$\sup_{\bar{v}} \left(\psi(\bar{v}) \dot{-} \alpha(\varphi(\bar{v}))\right) = 0.$

Note that if we build formulas by using the definition above it might give rise to uncountably many formulas even in a countable language. To avoid this, Stone - Weierstrass Theorem (See [3], p. 5) provides a countable dense set of connectives, so that we can approximate *any* formula to within any ϵ by some formula in this dense collection.

Common connectives we may use, by arity:

- Constants in $[0, 1]$.
- $\neg x = 1 - x$ and $\frac{1}{2}x = x/2$.
  Note that $\neg$ is not a negation, because $0 \mapsto 1$ and $\epsilon \mapsto 1 - \epsilon \neq 0$ (it does not turn false into true).
- $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $x \dot{-} y = (x - y) \vee 0$, $x \dot{+} y = (x + y) \wedge 1$, $|x - y|$.
  Here, $\wedge$ is "*or*" and $\vee$ is "*and*".

Generated from the set $\{\neg, \dot{-}\}$:

- $x \wedge y = x \dot{-} (x \dot{-} y)$
- $x \vee y = \neg(\neg x \wedge \neg y)$
- $x \dot{+} y = \neg(\neg x \dot{-} y)$
- $|x - y| = (x \dot{-} y) \vee (y \dot{-} x) = (x \dot{-} y) \dot{+} (y \dot{-} x)$

For example, let us show that $x \wedge y = x \dot{-} (x \dot{-} y)$. Take $a \in [0, 1]$:

- Consider $x(a) \leq y(a)$
  - $(x \wedge y)(a) = x(a)$
  - $(x \dot{-} (x \dot{-} y))(a) = x(a) - 0 = x(a)$
- Consider $x(a) > y(a)$
  - $(x \wedge y)(a) = y(a)$
  - $(x \dot{-} (x \dot{-} y))(a) = x(a) - (x(a) - y(a)) = y(a)$

We will usually use this set which has the advantage of being not only finite but also because it generates a set of connectives which is dense in the set of all functions $\{f : [0,1]^n \to [0,1]\}$ given the compact-open topology (see [3], p.5).

Free and *bound* occurrences of variables in L-formulas are defined in a manner similar to how this is done in classical logic, with the role of quantifiers played by sup and inf.

Definition. An *L-sentence* is an L-formula with no free variables.

As in classical logic, by writing a term t as $t(\vec{v})$ we mean that all variables occurring in t appear in $\vec{v}$. Similarly, for a formula φ the notation $\varphi(\vec{v})$ means that the tuple $\vec{v}$ contains all free variables of φ.

Also, let $\mathcal{M}$ be a L-structure with underlying 1-bounded metric space (M, d). Let A be a subset of M. Again, as in classical logic, we can extended L to a language $L(A)$ in the same way we did in chapter 2.

Now, for each $L(M)$-sentence σ, we define *the value of σ in $\mathcal{M}$*. This value is a real number in the interval $[0,1]$. The following definition is by induction on formulas.

Definition. ([2],Definition 3.3,p.16-17).

- $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for any t_1 and t_2 without free variables;
- for any n-ary relation symbol P of L and any $t_1, ..., t_n$ without free variables,

$$(P(t_1, ..., t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, ..., t_n^{\mathcal{M}});$$

- for any $L(M)$-sentences $\sigma_1, ..., \sigma_n$ and any continuous $u : [0,1]^n \to [0,1]$,

$$(u(\sigma_1, ..., \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, ..., \sigma_n^{\mathcal{M}});$$

- for any $L(M)$-formula $\varphi(v)$,

$$(\sup_v \varphi(v))^{\mathcal{M}} \text{ is the supremum in } [0,1] \text{ of the set } \{\varphi(a)^{\mathcal{M}} : a \in M\};$$

- for any $L(M)$-formula $\varphi(v)$,

$(\inf_v \varphi(v))^{\mathcal{M}}$ is the infimum in $[0,1]$ of the set $\{\varphi(a)^{\mathcal{M}} : a \in M\}$;

Remark. It is expected that given an $L(M)$-formula $\varphi(v_1, ..., v_n)$, we have that $\varphi^{\mathcal{M}}$ denotes the function from M^n to $[0,1]$ defined by

$$\varphi^{\mathcal{M}}(a_1, ..., a_n) = (\varphi(a_1, ..., a_n))^{\mathcal{M}}.$$

Remember that in classical logic we defined the value of L-sentence σ and this took its value in $\{0,1\}$. Analogously, we define the value of a L-condition E taking its value in $[0,1]$. First we give the definition of a L-condition E.
An *L-condition* E is a formal expression of the form $\varphi = 0$, where φ is an L-formula. We say that E is *closed* if φ is a sentence. Since any real number $r \in [0,1]$ is a connective, expressions of the form $\varphi = r$, $\varphi \leq r$ and $\varphi \geq r$ are conditions for any L-formula and $r \in [0,1]$. This is because we can consider the formulas $|\varphi - r|$, $\varphi \dot{-} r$ and $r \dot{-} \varphi$, respectively.

If E is the $L(M)$-condition $\varphi(v_1, ..., v_n) = 0$ and $a_1, ..., a_n$ are in M, we say E is *true* of $a_1, ..., a_n$ *in* $\mathcal{M}$ and write $\mathcal{M} \models E[a_1, ..., a_n]$ if $\varphi^{\mathcal{M}}(a_1, ..., a_n) = 0$.

Let us see a interesting fact. Quoting Sylvia Carlisle's words in [5]:

> The $\sup_v$ quantifier does act like the universal quantifier $\forall v$, since $(\sup_v \varphi(v))^{\mathcal{M}} = 0$ is true if and only if $\varphi(a)^{\mathcal{M}} = 0$ for all $a \in M$. However, the $\inf_v$ quantifier does not behave exactly like the existential quantifier $\exists v$. The condition $\inf_v \varphi(v) = 0$ only guarantees the existence of a sequence of elements making $\varphi(v)$ arbitrarily small (i.e. $\forall n \exists a_n \in \mathcal{M}$ such that $\varphi^{\mathcal{M}}(a_n) < 1/n$). In case we work in a compact structure then indeed there exists a convergent subsequence and thus, there exists $a \in M$ such that $\varphi(a) = 0$ (recall φ is continuous). The existence of a is guaranteed in some other classes of structures, e.g., ω-saturated structures, and discrete structures arising from classical logic.

Formulas in continuous logic define uniformly continuous functions whose moduli of uniform continuity depend only on the language L. The next theorem states this fact ([2],p. 17):

Theorem 3.2. *Let $t(v_1, ...v_n)$ be an L-term and $\varphi(v_1, ...v_n)$ an L-formula. Then there exist functions Δ_t and Δ_φ from $(0,1]$ to $(0,1]$ such that for any L-structure $\mathcal{M}$, Δ_t is a modulus of* *uniform continuity for the function* $t^{\mathcal{M}} : M^n \to M$ *and* Δ_φ *is a modulus of uniform continuity for the relation* $\varphi^{\mathcal{M}} : M^n \to [0,1]$.

Proof. The proof is by induction on the complexity of terms and then induction on the complexity of formulas. In the case of terms, this is an inductive argument using the fact that a composition of uniformly continuous mappings is uniformly continuous (see [3], Prop. 2.4). In the case of formulas one needs two more facts. First, all connectives are uniformly continuous as continuous mappings on a compact space. Second, if $\varphi(\bar{x}) = \inf_y \psi(y,\bar{x})$ or $\varphi(\bar{x}) = \sup_y \psi(y,\bar{x})$ then any uniform continuity modulus that $\psi(y,\bar{x})$ respects with respect to $\bar{x}$ is also respected by φ.

□

3.3 BASIC CONTINUOUS MODEL THEORY

Let L be a language for metric structures.

Definition. ([2],p.19-20) An L*-theory* is a set of closed L-conditions. If T is an L-theory and $\mathcal{M}$ is an L-structure, we say that $\mathcal{M}$ is a *model* of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models E$ for every closed condition $E \in T$.
If $\mathcal{M}$ is an L-structure, then $Th(\mathcal{M})$ denotes the *theory of* $\mathcal{M}$, i.e., the set of all closed L-conditions which are true in $\mathcal{M}$. When T is a theory of this kind, T will be said to be *complete*.
Also, let $\mathcal{C}$ be a class of L-structures, we say that $\mathcal{C}$ is *axiomatizable* if it contains all the models of $Th(\mathcal{C})$.

Example.
Let L be the language associated to probability structures. The following L-conditions are true in all probability structures. (see [2], p.94 for more details).

(1) Boolean algebra axioms:
$\sup_x (d(x \cap x, x)) = 0$;
$\sup_x \sup_y (d(x \cap y, y \cap x)) = 0$;
$\sup_x \sup_y (d((x \cap y) \cup y, y)) = 0$;
$\sup_x \sup_y \sup_z (d((x \cap y) \cap z, x \cap (y \cap z))) = 0$;

$\sup_x (d(x \cup x, x)) = 0$;
$\sup_x \sup_y (d(x \cup y, y \cup x)) = 0$;
$\sup_x \sup_y (d((x \cup y) \cap y, y)) = 0$;
$\sup_x \sup_y \sup_z (d((x \cup y) \cup z, x \cup (y \cup z))) = 0$;
$\sup_x \sup_y \sup_z (d(x \cap (y \cup z), (x \cap y) \cup (x \cap z))) = 0$;
$\sup_x \sup_y \sup_z (d(x \cup (y \cap z), (x \cup y) \cap (x \cup z))) = 0$;
$\sup_x (d(x \cup x^c, 1)) = 0$; (here 1 is the event of measure one)
$\sup_x (d(x \cap x^c, 0)) = 0$; (here the first 0 is the event of measure zero.)

(2) Measure axioms:
$\mu(0) = 0$ and $\mu(1) = 1$;
$\sup_x \sup_y (\mu(x \cap y) \dot{-} \mu(x)) = 0$;
$\sup_x \sup_y (\mu(x) \dot{-} \mu(x \cup y)) = 0$;
$\sup_x \sup_y |(\mu(x) - \mu(x \cap y)) - (\mu(x \cup y) - \mu(y))| = 0$.
The last axiom means that $\mu(x \cup y) + \mu(x \cap y) = \mu(x) + \mu(y)$ for all x, y, and it is well written thanks to the previous two axioms.

(3) Connection between d and μ:
$\sup_x \sup_y |d(x, y) - \mu(x \triangle y)| = 0$ where $x \triangle y$ denotes the Boolean term giving the symmetric difference: $(x \cap y^c) \cup (x^c \cap y)$.

We denote the set of L-conditions above by PrA.

3.4 ULTRAPRODUCTS

In this section we use ultraproducts to construct a new metric structure by getting the quotient of the direct product of a family of them. Note here that the new space looks like the old ones, i.e., they share the same properties. Furthermore, note this section is strongly based in [2], section 5, and hence we quote its definitions and examples verbatim, because we found no better wording.

Ultralimits

Let X be a topological space, and let $(x_i : i \in I)$ be an indexed family of elements of X. If U is an ultrafilter on I and $x \in X$, we write

$$\lim_{i,U} x_i = x$$

and say that x is the *U-limit* of $(x_i)_{i\in I}$ if for every open neighbourhood O of x, the set $\{i \in I : x_i \in O\} \in U$. A topological space X is Hausdorff compact if and only if for every indexed family $\{x_i : i \in I\}$ in X and every ultrafilter U on I, the U-limit of $(x_i : i \in I)$ exists and is unique. See [4].

Ultraproducts of bounded metric spaces

Let $((M_i, d_i) : i \in I)$ be a family of bounded metric spaces with diameter $\leq K$ for some fixed constant K. Let U be and ultrafilter on I. Then we can define a function $d : \prod_{i\in I} M_i \times \prod_{i\in I} M_i \to [0, K]$ by

$$d(x, y) = \lim_{i,U} d_i(x_i, y_i).$$

(Since for all $i \in I$, $d_i(x_i, y_i) \in [0, K]$ which is a compact Hausdorff space, so the U-limit exists.) Then d is a pseudometric on the cartesian product of the M_i. We now proceed to take the quotient metric space induced by this pseudometric: for $x, y \in \prod_{i\in I} M_i$, define $x \sim_U y$ to mean that $d(x, y) = 0$. Then $\sim_U$ is an equivalence relation. We define the *ultraproduct of* $((M_i, d_i) : i \in I)$ by writing

$$\left(\prod_{i\in I} M_i\right)_U := \left(\prod_{i\in I} M_i\right) / \sim_U .$$

Then d induces a metric on $(\prod_{i\in I} M_i)_U$, which we will denote by d. The bounded metric space $(\prod_{i\in I} M_i)_U$ (equipped with the metric d) is called the *U-ultraproduct* of the family $((M_i, d_i) : i \in I)$. The equivalence class of $(x_i : i \in I)$ is denoted by $(x_i : i \in I)_U$. If every M_i is complete, then $\left(\prod_{i\in I} M_i\right)_U$ is complete (see [2], Prop. 5.3).

Also, if $(M_i, d_i) = (M, d)$ for all $i \in I$, then the resulting U-ultraproduct is called the *U-ultrapower* of M and is denoted by $(M)_U$. In this situation, the map $T : M \to (M)_U$ defined by $T(x) = (x_i : i \in I)_U$, where $x_i = x$ for all $i \in I$, is an isometric embedding and is called the *diagonal embedding of M into* $(M)_U$.

A particular case of importance is the U-ultrapower of a compact metric space (M, d). In that case the diagonal embedding of M into $(M)_U$ is surjective as we show below:

Claim. Let M be a compact metric. Then $M_U \cong M$.
In fact, since $T : M \to M_U$ is a embedding, we must prove that it is surjective only. Let $(x_i)_{i\in I}$ be a representant of $((x_i)_{i\in I})_U$ and let x be its limit. So, $\lim\limits_{i,U} x_i = x \Leftrightarrow \forall\epsilon > 0\{i \in I : d(x_i, x) < \epsilon\} \in U \Leftrightarrow \forall\epsilon > 0\{i \in I : |d(x_i, x) - 0| < \epsilon\} \in U \Leftrightarrow \lim\limits_{i,U} d(x_i, x) = 0$. Then, $d((x_i)_{i\in I}, (x)_{i\in I}) = \lim_{i,U} d(x_i, x) = 0$. Thus, $T(x) = ((x_i)_{i\in I})_U$.

Ultraproducts of functions

Let $((M_i, d_i) : i \in I)$ and $((M'_i, d'_i) : i \in I)$ be two families of bounded metric spaces, all with diameter $\leq K$ for a fixed constant K. Let $(f_i : i \in I)$ be a family of n-ary functions with $f_i : M_i^n \to M'_i$ uniformly continuous for all $i \in I$. Suppose further that there is a single modulus of uniform continuity Δ for all the functions f_i. Let U be an ultrafilter on I. Then we can define an *ultraproduct function*

$$\left(\prod_{i\in I} f_i\right)_U : \left(\left(\prod_{i\in I} M_i\right)_U\right)^n \to \left(\prod_{i\in I} M'_i\right)_U$$

in the following manner: If $(x_i^k : i \in I) \in \prod_{i\in I} M_i$ for $k = 1, ..., n$, define

$$\left(\prod_{i\in I} f_i\right)_U \Big((x_i^1 : i \in I)_U, ..., (x_i^n : i \in I)_U\Big) = \Big(f_i(x_i^1, ..., x_i^n) : i \in I\Big)_U.$$

We claim that this is well-defined and is a uniformly continuous function that also has Δ as its modulus of uniform continuity.
In fact, for simplicity take $n = 1$ and fix $\epsilon > 0$. Suppose the distance between $((x_i)_{i\in I})_U$ and $((y_i)_{i\in I})_U$ in the ultraproduct $\left(\prod_{i\in I} M_i\right)_U$ is $< \Delta(\epsilon)$. There must exist $A \in U$ such that for all $i \in A$, $d_i(x_i, y_i) < \Delta(\epsilon)$. Since Δ is a modulus of uniform continuity for all the functions f_i, it follows that $d'_i(f_i(x_i), f_i(y_i)) \leq \epsilon$ for all $i \in A$. Hence the distance in the ultraproduct $\left(\prod_{i\in I} M'_i\right)_U$ between $((f_i(x_i))_{i\in I})_U$ and $((f_i(y_i))_{i\in I})_U$ must be $\leq \epsilon$.
Since ϵ is arbitrary, this shows good definition as well.

Ultraproducts of L-structures
Let $(\mathcal{M}_i : i \in I)$ be a family of L-structures and let U be an ultrafilter on I. Suppose

the underlying metric space of $\mathcal{M}_i$ is (M_i, d_i). Since there is a uniform bound on the diameter of those spaces, we may form their U-ultraproduct. For each function symbol f of L, the functions $f^{\mathcal{M}_i}$ all have the same modulus continuity Δ_f. Therefore the U-ultraproduct of this family of functions is well defined. The same is true if we consider a predicate function P of L. Moreover, the functions $P^{\mathcal{M}_i}$ all have their values in $[0,1]$ and thus the U-ultraproduct of $(P^{\mathcal{M}_i} : i \in I)$ can be regarded as a $[0,1]$-valued function on $M = \left(\prod_{i\in I} M_i\right)_U$ (see the claim above).

For each predicate symbol P of L, the interpretation of P in $\mathcal{M}$ is given by the ultraproduct of functions

$$P^{\mathcal{M}} = \left(\prod_{i\in I} P^{\mathcal{M}_i}\right)_U$$

which maps M^n to $[0,1]$ (using the canonical isomorphism $[0,1]_U \cong [0,1]$). For each function symbol f of L, the interpretation of f in $\mathcal{M}$ is given by the ultraproduct of functions

$$f^{\mathcal{M}} = \left(\prod_{i\in I} f^{\mathcal{M}_i}\right)_U$$

whichs maps M^n to M. For each constant symbol c of L, the interpretation of c in $\mathcal{M}$ is given by

$$c^{\mathcal{M}} = \left((c^{\mathcal{M}_i})_{i\in I}\right)_U.$$

The discussion above shows that this defines $\mathcal{M}$ to be an L-structure. We call $\mathcal{M}$ the *U-ultraproduct of the family* $(\mathcal{M}_i : i \in I)$ and denote by

$$\mathcal{M} = \left(\prod_{i\in I} \mathcal{M}_i\right)_U.$$

If all of the L-structures $\mathcal{M}_i$ are equal to the same structure $\mathcal{M}_0$, then $\mathcal{M}$ is called the *U-ultrapower of* $\mathcal{M}_0$ and is denoted by $(\mathcal{M}_0)_U$.

The next theorem is the analogue in this setting of Theorem 2.2 (it is proved in the page 12 of this text). This is sometimes known as the *Fundamental Theorem of Ultraproducts*.

Of course, the lemma 2.2 is valid here as well, but using U-limits instead.

Theorem 3.3. *([2], Theorem 5.4)(Łoś's Theorem for continuous logic)*
Let $(\mathcal{M}_i : i \in I)$ be a family of L-structures. Let U be any ultrafilter on I and let $\mathcal{M}$ be the U-ultraproduct of $(\mathcal{M}_i : i \in I)$. Let $\varphi(v_1, ..., v_n)$ be an L-formula. If $a_k = \big((a_i^k)_{i\in I}\big)_U$ are elements of $\mathcal{M}$ for $k = 1, ..., n$, then

$$\varphi^{\mathcal{M}}(a_1, ..., a_n) = \lim_{i,U} \varphi^{\mathcal{M}_i}(a_i^1, ..., a_i^n).$$

Proof. By induction on the complexity of φ and the commutativity of U-limit with continuous functions.
If $\varphi(v_1, ..., v_n)$ is atomic, we can assume that $\varphi(\bar{v}) = P(t_1(\bar{v}), ..., t_m(\bar{v}))$. For any given $a_1, ..., a_n \in \mathcal{M}$ we have

$$\begin{aligned}\varphi^{\mathcal{M}}(a_1, ..., a_n) &= P(t_1, ...t_m)^{\mathcal{M}}(a_1, ..., a_n)\\ &= \lim_{i,U} P(t_1, ..., t_m)^{\mathcal{M}_i}(a_1, ..., a_n)\\ &= \lim_{i,U} P^{\mathcal{M}_i}(t_1^{\mathcal{M}_i}(a_i^1, ..., a_i^n), ..., t_m^{\mathcal{M}_i}(a_i^1, ..., a_i^n))\\ &= \lim_{i,U} \varphi^{\mathcal{M}_i}(a_i^1, ..., a_i^n).\end{aligned} \tag{1}$$

If the result holds for $\varphi_1(v_1, ..., v_n), ..., \varphi_m(v_1, ..., v_n)$ and $u : [0, 1]^m \to [0, 1]$ is a uniformly continuous function, then for all $a_1, ..., a_n \in \mathcal{M}$ we have

$$\begin{aligned}u^{\mathcal{M}}(\varphi_1(a_1, ..., a_n), ..., \varphi_m(a_1, ..., a_n)) &= u(\varphi_1^{\mathcal{M}}(a_1, ..., a_n), ..., \varphi_m^{\mathcal{M}}(a_1, ..., a_n))\\ &= u(\lim_{i,U} \varphi_1^{\mathcal{M}_i}(a_i^1, ..., a_i^n), ..., \lim_{i,U} \varphi_m^{\mathcal{M}_i}(a_i^1, ..., a_i^n))\\ &= \lim_{i,U} u(\varphi_1^{\mathcal{M}_i}(a_i^1, ..., a_i^n), ..., \varphi_m^{\mathcal{M}_i}(a_i^1, ..., a_i^n))\\ &= \lim_{i,U} u^{\mathcal{M}_i}(\varphi_1(a_i^1, ..., a_i^n), ..., \varphi_m(a_i^1, ..., a_i^n))\end{aligned} \tag{2}$$

Now, let us consider the case $\sup_{\bar{v}} \psi(\bar{v}, \bar{y})$ and suppose the hypothesis holds for ψ. To make the notations simpler we assume $\bar{y} = \emptyset$. Then for $a_1, ..., a_n \in \mathcal{M}$ we have

$$\psi^{\mathcal{M}}(a_1, ..., a_n) = \lim_{i,U} \psi^{\mathcal{M}_i}(a_i^1, ..., a_i^n) \leq \lim_{i,U}(\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}}$$

So,

$$(\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}} \leq \lim_{i,U} (\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}_i}.$$

For the converse, let $r = (\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}}$ and suppose the inequality above is strict. Take r' such that $r < r' < \lim_{i,U} (\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}_i}$. Then for U-almost all i we have $r' < (\sup_{\bar{v}} \psi(\bar{v}))^{\mathcal{M}_i}$. So, for U-almost all i there exists $b_i = (b_i^1, ..., b_i^n)$ such that $r' < \psi^{\mathcal{M}_i}(b_i^1, ..., b_i^n)$. This means that $r < \psi^{\mathcal{M}}(b_1, ..., b_n)$ which is a contradiction. Thus we see that the equality holds. The case of inf is similar.

□

Corollary 3.4. *([2], Corollary 5.8)(Compactness Theorem for continuous logic)*
Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T is finitely satisfiable in $\mathcal{C}$ [1]. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.

Proof. This is in every way the same that for classical logic (proposition 2.3). We use conditions instead of sentences, and recall that conditions have the form $\varphi = 0$, but $\lim_{i,U} 0 = 0$.

□

The finite satisfiability hypothesis can be weakened to an approximate version, as follows:

Definition. For any set Σ of L-conditions, let Σ^+ be the set of all conditions $\varphi \leq 1/n$ such that $\varphi = 0$ is an element of Σ and $n \geq 1$.

Corollary 3.5. *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T^+ is finitely satisfiable in $\mathcal{C}$. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

Proof. Just note that T and T^+ have the same models.

□

1 We call T finitely satisfiable in $\mathcal{C}$ if every finite subset of T is satisfiable in $\mathcal{C}$.

3.5 RELATION BETWEEN ULTRAPRODUCTS

In this section let us show there exists a natural isomorphism between the first and second ultraproduct, on classical structures given the discrete metric.
We write $(\mathcal{A}_i : i \in I)$ for a family of classical structures, and for each $i \in I$ let $\mathcal{M}_i$ be the metric structure which arises from $\mathcal{A}_i$ given the discrete metric, as discussed on page 16.

$$\begin{array}{ccc} \textit{classical logic} & & \textit{discrete metric logic} \\ \Phi : \Pi\mathcal{A}_i/U & \longrightarrow & \Pi\mathcal{M}_i/U \\ a_U & \mapsto & \Phi(a_U) = a_U \end{array}$$

Since $a = b \Leftrightarrow d(a,b) = 0$, the equivalence relations $\sim_U$ in either setting are the same, and so there is a natural way to identify a element of $\Pi\mathcal{A}_i/U$ into $\Pi\mathcal{M}_i/U$. That is what $\Phi(a_U) = a_U$ means.

First, we need to show this is well-defined.

$$\begin{aligned} & a \sim_U b \Leftrightarrow \{i \in I : a_i = b_i\} \in U \\ \Longrightarrow\ & \{i \in I : d(a_i, b_i) = 0\} \in U \\ \Longrightarrow\ & \forall \epsilon > 0 \{i \in I : d(a_i, b_i) < \epsilon\} \in U \\ \Longrightarrow\ & \lim_{i,U} d(a_i, b_i) = 0 \\ \Longrightarrow\ & d((a_i), (b_i)) = 0 \\ \Longrightarrow\ & a \sim_U b. \end{aligned}$$

Injective:

$$\begin{aligned} & \lim_{i,U} d(a_i, b_i) = 0 \\ \Longrightarrow\ & \forall \epsilon > 0 \{i \in I : d(a_i, b_i) < \epsilon\} \in U \\ \Longrightarrow\ & \{i \in I : d(a_i, b_i) < 1/2\} \in U \\ \Longrightarrow\ & \{i \in I : d(a_i, b_i) = 0\} \in U \text{ (because } d(a_i, b_i) \text{ equals either 0 or 1)} \\ \Longrightarrow\ & \{i \in I : a_i = b_i\} \in U \\ \Longrightarrow\ & a \sim_U b. \end{aligned}$$

Surjective:

Given a_U in the discrete metric construction, we have $a \in \prod_{i\in I} A_i$ and form a_U by using classical logic construction.

We note that the atomic formulas are the same in both classical logic and metric settings because constants, functions and terms are defined exactly the same way, and

$$R^{\mathcal{M}_i}(a_1, ..., a_n) = \begin{cases} 0 & \text{if} \quad (a_1, ..., a_n) \in R^{\mathcal{A}_i} \\ 1 & \text{if} \quad (a_1, ..., a_n) \notin R^{\mathcal{A}_i} \end{cases}$$

(uniformly continuous because the metric is discrete).
We see that for an atomic formula $\alpha(v_1, ...v_n)$, we have

$$\alpha^{\prod_U \mathcal{A}_i}(a_1, ..., a_n) = \begin{cases} 0 & \text{if} \quad \prod_U \mathcal{A}_i \models \alpha(a_1, ..., a_n) \\ 1 & \text{if} \quad \prod_U \mathcal{A}_i \not\models \alpha(a_1, ..., a_n) \end{cases}$$

So, Φ is an isomorphism (it preserves and reflects satisfaction). Of course, the word "isomorphism" must be taken lightly here, since the languages and the logics themselves are different.

We can understand this section under the view of functor maps.
First of all, let $\mathcal{C}$ be the collection of structures for a language L in classical logic, i.e., $\mathcal{C} = \{\mathcal{A} : \mathcal{A} \textit{ is a L-structure}\}$ and let $\mathcal{D}$ be the collection of metric structures for the corresponding language L' in continuous logic, i.e., $\mathcal{D} = \{\mathcal{M} : \mathcal{M} \textit{ is a } L'\textit{-structure}\}$. Note the language L' is obtained from L in the way we describe on page 17, but D is not confined to discretized classical structures.

The conversion we provide in this section is a function $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, called functor and it also preserves homomorphisms (a homomorphism between two structures in $\mathcal{C}$ will induce a homomorphism between their images in $\mathcal{D}$).

To work with an ultraproduct, say using a ultrafilter U on a set I, form for each collection $\mathcal{C}$ of structures, the collection $\mathcal{C}^I$ of sequences of structures in $\mathcal{C}$ indexed by I.

Then the ultraproduct construction is a functor $\mathcal{F}_{\mathcal{C}} : \mathcal{C}^I \mapsto \mathcal{C}$. Ultrapowers precompose $\mathcal{F}_{\mathcal{C}}$ with the diagonal functor $\Delta_{\mathcal{C}} : \mathcal{C} \mapsto \mathcal{C}^I$.

4 ON BANACH SPACES

In this chapter we do not intend to set the theory of Banach spaces. We consider that these concepts and definitions are familiar for the reader. Thus, we start with the construction of ultraproducts in the mentioned spaces and take [8] as our source for this section. Thus, our role to play here is to justify some statements claimed there.

4.1 ULTRAPRODUCTS IN BANACH SPACES

Let $(X_i)_{i\in I}$ be a family of Banach spaces. Consider the set

$$\ell_\infty(I, X_i) = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sup_{i\in I} \|x_i\| < \infty \right\}$$

Claim. The set $\ell_\infty(I, X_i)$ together with the norm $\|(x_i)\| = \sup_{i\in I}\|x_i\|$ is a Banach space under componentwise addition and scalar multiplication, we mean, $(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$ and $\lambda(x_i)_{i\in I} = (\lambda x_i)_{i\in I}$.

Proof. In fact, if $(\vec{x}_n) = ((x_{n,i}))$ is a Cauchy sequence in $l_\infty(I, X_i)$, then for any $\epsilon > 0$, there is an $N_\epsilon \in \mathbb{N}$ so for all $n, m \geq N_\epsilon$, $\|\vec{x}_n - \vec{x}_m\| = \sup_{i\in I}\|x_{n,i} - x_{m,i}\| < \epsilon$. Thus in each component, i.e., for each $i \in I$, $(x_{n,i})$ is a Cauchy sequence in X_i, which is a Banach space, so it has a limit x_i. Note that for all $n \geq N_1$, $\|\vec{x}_n - \vec{x}_{N_1}\| < 1$, so for all $i \in I$, $\|x_{n,i} - x_{N_1,i}\| < 1$. Fix an index $i \in I$. Since $(x_{n,i}) \to x_i$ there is an $M_{i,1} \in \mathbb{N}$ such that for all $n \geq M_{i,1}$, $\|x_i - x_{n,i}\| < 1$. Now for $n \geq \max(N_1, M_{i,1})$, we have

$$\|x_i\| \leq \|x_i - x_{n,i}\| + \|x_{n,i}\| \leq 1 + \|\vec{x}_n\| \leq 1 + \|\vec{x}_n - \vec{x}_{N_1}\| + \|\vec{x}_{N_1}\| \leq 2 + \|\vec{x}_{N_1}\| < \infty.$$

Thus, $\sup_{i\in I}\|x_i\|_{X_i} \leq 2 + \|\vec{x}_{N_1}\| < \infty$. Then consider $\vec{x} = (x_i)$. We have shown that $\vec{x} \in l_\infty(I, X_i)$. We claim $\vec{x}_n \to \vec{x}$. Given $\epsilon > 0$, then for $n \geq N_{\epsilon/2}$ we claim

$$\|\vec{x}_n - \vec{x}\| = \sup_{i\in I}\|x_{n,i} - x_i\| < \epsilon.$$

As before for all $n \geq N_{\epsilon/2}$, $\|\vec{x}_n - \vec{x}_{N_{\epsilon/2}}\| < \epsilon/2$, so for all $i \in I$, $\|x_{n,i} - x_{N_{\epsilon/2},i}\| < \epsilon/2$. Fix an index $i \in I$. Since $(x_{n,i}) \to x_i$ there is an $M_{i,\epsilon/2} \in \mathbb{N}$ such that for all $n \geq M_{i,\epsilon/2}$, $\|x_i - x_{n,i}\| < \epsilon/2$.

For each $i \in I$, let $m_i = \max(N_{\epsilon/2}, M_{i,\epsilon/2})$, then for all $i \in I$, $n \geq N_{\epsilon/2}$.

$$\|x_{n,i} - x_i\| \leq \|x_{n,i} - x_{m_i,i}\| + \|x_{m_i,i} - x_i\| < \|\vec{x}_n - \vec{x}_{m_i}\| + \epsilon/2 < \epsilon/2 + \epsilon/2 < \epsilon.$$

Thus $\ell_\infty(I, X_i)$ is a Banach space.

□

For U an ultrafilter on I, let $N_U = \{(x_i)_{i\in I} \in \ell_\infty(I, X_i) : \lim_{i,U} \|x_i\| = 0\}$.

Remark. $\lim_{i,U} \|x_i\| = 0 \Leftrightarrow \forall \epsilon > 0 \{i \in I : \|x_i\| < \epsilon\} \in U$.

Claim. N_U is a closed linear subspace of $\ell_\infty(I, X_i)$.

Proof. In fact, suppose $(x_i), (y_i) \in N_U$, $\lambda \neq 0$. Then for a given $\epsilon > 0$, we have

$$\{i \in I : \|\lambda x_i\| < \epsilon\} = \{i \in I : \|x_i\| < \tfrac{\epsilon}{|\lambda|}\} \in U$$

so $(\lambda x_i) \in N_U$ and

$$\{i \in I : \|x_i + y_i\| < \epsilon\} \supseteq \{i \in I : \|x_i\| < \epsilon/2\} \cap \{i \in I : \|y_i\| < \epsilon/2\} \in U.$$

Thus $(x_i + y_i) \in N_U$.

Furthermore, N_U is closed: To prove this, let us show that $\ell_\infty(I, X_i) \setminus N_U$ is open. Take any $(x_i)_{i\in I} \in \ell_\infty(I, X_i) \setminus N_U$. We note that $\|(x_i)\| = \sup_{i\in I} \|x_i\| = M$, so we have $\|x_i\| \in [0, M] \subseteq \mathbb{R}$ that is compact Hausdorff, thus $\lim_{i,U} \|x_i\|$ exists and

$$\lim_{i,U} \|x_i\| = L > 0 \text{ because } (x_i) \notin N_U.$$

We claim $B\left((x_i), \frac{L}{2}\right) \subseteq \ell_\infty(I, X_i) \setminus N_U$.
Take any $(y_i) \in B\left((x_i), \frac{L}{2}\right)$, then $\|(y_i) - (x_i)\| = \sup_{i\in I} \|y_i - x_i\| < \frac{L}{2}$, then $\|y_i - x_i\| < \frac{L}{2}$ for all $i \in I$.

Now, $\lim_{i,U} \|x_i\| = L > 0 \Leftrightarrow \forall \epsilon > 0 \{i \in I : \|x_i\| \in (L - \epsilon, L + \epsilon)\} \in U \Leftrightarrow$
$\forall \epsilon > 0 \{i \in I : |L - \|x_i\|| < \epsilon\} \in U$.

Taking $\epsilon = L$ and for $i \in I_{\frac{L}{2}} = \{i \in I : |L - \|x_i\|| < \frac{L}{2}\} \in U$, we have

$$\|y_i\| \geq \|x_i\| - \|y_i - x_i\| \geq \frac{3L}{4} - \frac{L}{2} = \frac{L}{4}$$

From this, we have $\{i \in I : \|y_i\| < \frac{L}{4}\} \subseteq I \setminus I_{\frac{L}{2}} \notin U$.
So, $\lim_{i,U} \|y_i\| \neq 0$. Thus $(y_i) \notin N_U$ as claimed.

□

Finally, we can define the *ultraproduct of* $(X_i)_{i \in I}$ *with respect to* U by

$$(X_i)_U = \ell_\infty(I, X_i)/N_U$$

We will denote the equivalence class of $(x_i) \in \ell_\infty(I, X_i)$ in the ultraproduct as $(x_i)_{i \in I} + N_U$.
We equip the ultraproduct $(X_i)_U$ with the canonical quotient norm:

$$\|(x_i)_{i \in I} + N_U\| = \inf_{(a_i) \in N_u} \|(x_i - a_i)\| = \inf_{(a_i) \in N_u} \sup_{i \in I} \|x_i - a_i\|.$$

Claim. The norm can be computed as

$$\|(x_i)_{i \in I} + N_U\| = \lim_{i,U} \|x_i\|.$$

Proof. In fact, let $L = \lim_{i,U} \|x_i\|$. Now, we note that for any $\epsilon > 0$, we have $I_\epsilon = \{i \in I : |\|x_i\| - L| < \epsilon\} \in U$.
Define the sequence (α_i) by:

$$\alpha_i = \begin{cases} x_i & \text{if} \quad i \notin I_\epsilon \\ 0 & \text{if} \quad i \in I_\epsilon \end{cases}$$

Note that $\{i \in I : \alpha_i = 0\} \supseteq I_\epsilon \in U$. Thus $\{i \in I : \alpha_i = 0\} \in U$.
Also, we have

$$\lim_{i,U} \|\alpha_i\| = 0$$

because $(\alpha_i)_{i \in I}$ is U-almost always null. Thus $(\alpha_i) \in N_U$ and for all $i \in I$,

$$\|x_i - \alpha_i\| = \begin{cases} 0 & \text{if} \quad i \notin I_\epsilon \\ \|x_i\| & \text{if} \quad i \in I_\epsilon \end{cases} < L + \epsilon$$

Thus, $\|(x_i)_{i \in I} + N_U\| = \inf_{(a_i) \in N_U} \|(x_i - a_i)\| \leq \|(x_i - \alpha_i)\| = \sup_{i \in I} \|x_i - \alpha_i\| \leq L + \epsilon$.

Conversely, if $(a_i) \in N_U$,

$$I_{(a_i),\epsilon} = \{i \in I : \|a_i\|_{X_i} < \epsilon/2\} \cap \{i \in I : |\|x_i\| - L| < \epsilon/2\} \in U.$$

Then for all $i \in I_{(a_i),\epsilon}$, we have $\|x_i - a_i\| \geq \|x_i\| - \|a_i\| > L - \epsilon/2 - \epsilon/2 = L - \epsilon$. Thus for any $(a_i) \in N_U$,

$$\sup_{i \in I} \|x_i - \alpha_i\| \geq L - \epsilon.$$

So,

$$\|(x_i)_{i \in I} + N_U\| = \inf_{(a_i) \in N_U} \sup_{i \in I} \|x_i - a_i\| \geq L - \epsilon.$$

So, we have shown that for all $\epsilon > 0$,

$$\lim_{i,U} \|x_i\| - \epsilon \leq \|(x_i)_{i \in I} + N_U\| \leq \lim_{i,U} \|x_i\| + \epsilon$$

And we conclude that $\|(x_i)_{i \in I} + N_U\| = \lim_{i,U} \|x_i\|$ as claimed.

□

If all the spaces $X_i = X$ then we speak of the *ultrapower* $(X)_U$. There is a canonical isometric embedding E of X into its ultrapower $(X)_U$ which is defined by $E(x) = (x_i)_U$ where $x_i = x$ for all $i \in I$.

Now that we introduced the ultraproduct of Banach spaces, we can define the ultraproduct of operators. Let $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ be families of Banach spaces indexed by the same set I, and for each $i \in I$, let $T_i \in \mathcal{B}(X_i, Y_i)$ be a bounded linear map from X_i to Y_i, such that

$$\sup_{i \in I} \|T_i\| < \infty.$$

The *ultraproduct of the family of operators $(T_i)_{i \in I}$ with respect to the ultrafilter U on I is* $(T_i)_U$ defined by

$$(x_i)_{i \in I} + N_U \mapsto (T_i x_i)_{i \in I} + N_U.$$

We claim that this linear map is well-defined. If $\|T_i\| = 0$ for all $i \in I$, then $(T_i)_U((x_i)_U) = (0)_U = (T_i)_U((y_i)_U)$. If $0 < \sup_{i \in I} \|T_i\| = M < \infty$, and if $(x_i) \sim_U (y_i)$, then $\|(x_i - y_i)_U\| = \lim_{i,U} \|x_i - y_i\| = 0$, so for all $\epsilon > 0$, $\{i \in I : \|x_i - y_i\| < \epsilon\} \in U$, thus

$$\{i \in I : \|T_i(x_i - y_i)\| < \epsilon\} \supseteq \{i \in I : \|x_i - y_i\| < \tfrac{\epsilon}{M}\} \in U.$$

And in particular

$$\lim_{i,U}\|T_i(x_i - y_i)\| = \lim_{i,U}\|T_i(x_i) - T_i(y_i)\| = \|(T_i(x_i))_U - (T_i(y_i))_U\| = 0$$

So $(T_i(x_i))_U = (T_i(y_i))_U$.

We see that $(T_i)_U$ is a linear map. Moreover, we show that $\|(T_i)_U\| = \lim\limits_{i,U}\|T_i\|$.

Proof. In fact, it is clear $\lim\limits_{i,U}\|T_i\|$ exists, since $\sup_{i\in I}\|T_i\| = M < \infty$.
So, $\|T_i\|$ takes values in $[0, M]$ which is compact Hausdorff, so $\lim_{i,U}\|T_i\| = L$ exists.
Let $\epsilon > 0$ be given. Let $\|(x_i)_U\| = 1$, then

$$I_0 = \{i \in I : |1 - \|x_i\|| < \epsilon\} \subseteq \{i \in I : \|x_i\| < 1 + \epsilon\} \in U.$$

So we can pick an equivalent sequence $(x_i') \sim_U (x_i)$ with $\|x_i'\| < 1 + \epsilon$ for all $i \in I$. Now

$$\|T_i(x_i')\| \leq \|T_i\|\|x_i'\| \leq \|T_i\|(1 + \epsilon).$$

So in particular,

$$\|T((x_i)_U)\| = \lim_{i,U}\|T_i(x_i')\| \leq (1 + \epsilon)\lim_{i,U}\|T_i\|.$$

Likewise, for all $i \in I$ we can find an $x_i'' \in X_i$ with $\|x_i''\| > 1 - \epsilon$ and $\|T_i(x_i'')\| \geq \|T_i\|(1 - \epsilon)$. Thus

$$\|T((x_i))_U\| = \lim_{i,U}\|T_i(x_i')\| \geq \lim_{i,U}\|T_i\|(1 - \epsilon).$$

□

Having introduced the ultraproduct of Banach spaces, we shall now mention the following two results. We quote verbatim [14]:

Claim. *Any Banach space X is isometric to a subspace of some ultraproduct of its finite-dimensional subspaces.*

In words of [8], this means that if X is a Banach space and $\mathcal{B}$ is a family of Banach spaces such that for each $\epsilon > 0$, and each finite-dimensional subspace M of X there is a space $E = E_{M,\epsilon} \in \mathcal{B}$ such that M is $(1 + \epsilon)$-isomorphic [1] to a subspace of E. There exists an ultrafilter $\mathcal{U}$ on an index I and a map from I into $\mathcal{B}$ sending $i \mapsto E_i \in \mathcal{B}$.

1 Recall that an operator $T : E \to F$ is a $(1 + \epsilon)$-*isomorphism* $(0 < \epsilon < 1)$ if T is an isomorphism and for all x, $|\|Tx\| - \|x\|| \leq \epsilon\|x\|$

Proof. (A sketch only).
Let I be the set of finite dimensional subspaces of X, ordered by inclusion. We get a filter generated by all the upperset of any $M_0 \in I$ and thus an ultrafilter U containing this filter.
Let us denote $i \in I$ by (M_i, ϵ_i), and for each i there is a space $E_i \in \mathcal{B}$ and a $(1+\epsilon)$-isomorphism $T_i : M_i \to E_i$. Then, defining a map $F : X \to \prod_{\mathcal{U}} E_i$ for $x \in X$ by

$$F(x) = (y_i)_{\mathcal{U}}, y_i = \begin{cases} T_i(x) & \text{if} \quad x \in M_i \\ 0 & \text{if not} \end{cases}$$

we have the desired linear isometric.

□

To finish this section, we also mention that ultraproducts are used to characterize axiomatizability. The next result is thanks to C. W. Henson.

Proposition 4.1. *([2], Proposition 5.14)*
Suppose that $\mathcal{C}$ is a class of L-structures. The following statements are equivalent:

- *$\mathcal{C}$ is axiomatizable in L, that is, $\mathcal{C}$ is the class of structures which satisfy some given theory.*
- *$\mathcal{C}$ is closed under isomorphisms and ultraproducts, and its complement is closed under ultrapowers.*

Thus, we have the next result:

Claim. *The class of L_p Banach spaces, where $1 \leq p < \infty$, is axiomatizable.*

Proof. This fact was proved by C. Ward Henson. See ([2], pages $98 - 100$).

□

4.2 COMPARING ULTRAPRODUCTS

Now, let us compare the unit open ball of the ultraproduct of Banach spaces to the ultraproduct of metric structures.

Start with $(x_i) + N_U \in B_1(0) \subseteq (X_i)_U$.
Recall that
$(x_i) + N_U = \{(y_i)_{i\in I} : (x_i)_{i\in I} - (y_i)_{i\in I} \in N_U\}$
$= \{(y_i)_{i\in I} : \lim_U \|x_i - y_i\| = 0\}$
$= \{(y_i)_{i\in I} : \forall\epsilon > 0 \{i \in I : \|x_i - y_i\| < \epsilon\} \in U\}$.

Since $(x_i) + N_U \in B_1(0)$, we have that $\|(x_i) + N_U\| = \lim_{i,U} \|x_i\| = k$, for some $k < 1$.

Define $(z_i)_{i\in I} \in \prod_{i\in I} B_1^{X_i}(0)$:

$$z_i = \begin{cases} x_i & \text{if } \|x_i\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Take $\epsilon > 0$ small enough such that $k + \epsilon < 1$, so $\{i \in I : \|x_i\| < k + \epsilon\} \in U$, hence $z_i = x_i$ U-almost always.
So, we can form $z = (z_i)_U \in \left(\prod B_1^{X_i}(0)\right)_U$ and define $z = \Phi(x)$ where $x = (x_i)_U$.

$$\begin{array}{rcl} \textit{Banach space} & & \textit{continuous logic} \\ \Phi : (X_i)_U \supseteq B_1(0) & \longrightarrow & B_1(0) \subseteq \left(\prod B_1^{X_i}(0)\right)_U \\ x & \longmapsto & \Phi(x) = z \end{array}$$

Let us show that Φ is well-defined.

Take $(y_i)_{i\in I} \in (x_i)_{i\in I} + N_U$, then we define:

$$w_i = \begin{cases} y_i & \text{if } \|y_i\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and we must prove $z = w = (w_i)_U$.
In fact,
$d(z, w) = \lim_{i,U} d_i(z_i, w_i) = \lim_{i,U} d_i(x_i, y_i) = 0$ (because $z_i = x_i$ and $w_i = y_i$ U-almost always).

Now, we show that $\|z\| = \|x\|$

$\|z\| \overset{\text{Łoś}}{=} \lim\limits_{i,U} \|z_i\| = k$ (here the norm is considered as a predicate)
$\Leftrightarrow \forall \epsilon > 0 \{i \in I : |\|z_i\| - k| < \epsilon\} \in U$
$\Leftrightarrow \forall \epsilon > 0 \{i \in I : |\|x_i\| - k| < \epsilon\} \in U$ (because $z_i = x_i$ U-almost always)
$\Leftrightarrow k = \lim\limits_{i,U} \|x_i\|_{X_i} = \|x\|$.

Thus, $\|\Phi(x)\| = \|x\|$.

Injectivity:
$\Phi(x) = \Phi(y) \Leftrightarrow z = w \Leftrightarrow (z_i)_U = (w_i)_U \Leftrightarrow (z_i)_{i\in I} \sim_U (w_i)_{i \in I}$
$\Leftrightarrow d(z, w) = 0 \Leftrightarrow \lim_{i,U} d_i(z_i, w_i) = 0$
$\Leftrightarrow \forall \epsilon > 0 \{i \in I : d_i(z_i, w_i) < \epsilon\} \in U$
$\Leftrightarrow \forall \epsilon > 0 \{i \in I : d_i(x_i, y_i) < \epsilon\} \in U$
$\Leftrightarrow \forall \epsilon > 0 \{i \in I : \|x_i - y_i\| < \epsilon\} \in U$
$\Leftrightarrow \{i \in I : (x_i)_{i\in I} - (y_i)_{i \in I} \in N_U\}$
$\Leftrightarrow (x_i)_{i\in I} + N_U = (y_i)_{i\in I} + N_U \Leftrightarrow x = y$.

Surjectivity onto the open ball:
Take $w = (w_i)_U$ with every $w_i \in B_1^{X_i}(0)$, and assume $k = \|w\| < 1$.
Form $x = (w_i)_{i\in I} + N_U \in (X_i)_U$ in the ultraproduct of Banach spaces. Thus $\Phi(x) = w$ since every $\|w_i\| < 1$.
Let us show that $x \in B_1(0) \subseteq (X_i)_U$.
We have $x = (w_i)_{i\in I} + N_U = \{(z_i)_{i\in I} \in \prod_{i\in I} X_i : (w_i - z_i)_{i\in I} \in N_U\}$
$= \{(z_i)_{i\in I} \in \prod_{i\in I} X_i : \forall \epsilon > 0 \{i \in I : \|w_i - z_i\| < \epsilon\} \in U\}$.

So $\|x\| = \lim_{i,U} \|w_i\| = k$. Thus $x \in B_1(0)$.

Linearity:

Start with x, y in $B_1(0) \subseteq (X_i)_U$ and scalars $\alpha, \mu \in \mathbb{R}$ such that $|\alpha| + |\mu| \leq 1$, so $\alpha x + \mu y \in B_1(0)$.

Since $\alpha x + \mu y \in B_1(0)$, we have $\lim_{i,U} \|\alpha x_i + \mu y_i\| = k$, for some $k < 1$.

Define $(t_i)_{i\in I} \in \prod_{i\in I} B_1^{X_i}(0)$, thus:

$$t_i = \begin{cases} \alpha x_i + \mu y_i & \text{if } \|\alpha x_i + \mu y_i\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Taking ϵ small enough such that $k + \epsilon < 1$, we have $\{i \in I : \|\alpha x_i + \mu y_i\| < k + \epsilon\} \in U$, hence $t_i = \alpha x_i + \mu y_i = \alpha z_i + \mu w_i$ U-almost always. Note that $\Phi(\alpha x + \mu y) = (t_i)_U$.

Thus $\Phi(\alpha x + \mu y) = \alpha\Phi(x) + \mu\Phi(y)$.

5 ON URYSOHN SPACE

In this chapter we describe some properties of the Urysohn space in continuous logic. We focus on understanding the model-theoretic properties, and so we only demonstrate the first two and give references for the proofs of the rest of them.

Since we are working with metric spaces bounded by 1, our space will be called *Urysohn sphere* and denoted by $\mathcal{U}$.

Definition. The *Urysohn sphere* $\mathcal{U}$ is the unique (up to isometries) *universal* and *ω-homogeneous* complete separable metric space (bounded by 1). We mean:

- (*Universal*) Every separable metric space (bounded by 1) can be isometrically embedded into it.
- ($\omega - homogeneous$) Every isometry between finite subspaces can be extended to an isometry of the whole space onto itself.

5.1 CONSTRUCTION OF THE URYSOHN SPHERE

We show a sketch of its construction by using *Katětov maps* and inspired by [12].

A map $f : X \to [0,1]$ is a *Katětov map* if

$$\forall x, y \in X \; |f(x) - f(y)| \leq d(x,y) \leq f(x) + f(y).$$

Let (X, d) be our starting separable metric space (bounded by 1). It determines a metric space

$$E(X) = \{f : X \to [0,1] : |f(x) - f(y)| \leq d(x,y) \leq f(x) + f(y), \forall x, y \in X\}$$

endowed with the $sup-metric$ $(d(f,g) = \sup\{|f(x) - g(x)|: x \in X\} \forall f, g \in E(X))$. One can identify $x \in X$ with the function $f_x(z) : X \mapsto [0,1]$ where $f_x(z) = d(z,x)$, and thus we assume $X \subset E(X)$ ([13], Proposition 2.2).

If Y is a subspace of X, then $E(Y)$ embeds isometrically into $E(X)$ via the *Katětov extension*: every $f \in E(Y)$ extends to $\hat{f} \in E(X)$,

$$\hat{f}(x) = \inf\{d(x,y) + f(y) : y \in Y\}$$

Thus we may consider $E(Y)$ as a subspace of $E(X)$, and we can define for cardinals $\kappa > 1$,

$$E(X,\kappa) = \bigcup\{E(Y) : Y \subset X, 0 < |Y| < \kappa\} \subset E(X)$$

Now, taking $\kappa = \omega$ and by induction starting with $X_0 = X$, define $X_{i+1} = E(X_i, \omega)$. Let now $X_\omega = \bigcup_{i<\omega} X_i$; the construction ensures that X_ω has the extension property, i.e., for all $A \subseteq X_\omega$ finite and $f \in E(A)$, there is $x \in X_\omega$ such that $\forall a \in A$, $d(x,a) = f(a)$. Indeed, any finite subset $\{y_1, ..., y_n\}$ of X_ω is contained in X_m for some big enough m; then the extension to X_m of any map $f \in E(\{y_1, ..., y_n\})$ appears as an element of X_{m+1}, which shows that there is indeed a point $y \in X_\omega$ such that $d(y, y_i) = f(y_i)$ for all $i = 1, ..., n$. Hence, the completion of X_ω has the extension property too, and by ([13], Theorem 3.2) it is universal and ω-homogeneous. Thus, we have proved the existence of the Urysohn sphere.

5.2 MODEL THEORETIC PROPERTIES

These properties studied by Usvyatsov are found in [16]. Let us detail a little more.
Let us consider $\mathcal{U}$ as a metric structure in the empty language (containing only the distance function $d : \mathcal{U}^2 \to [0,1]$).
Denote by Φ_n the collection of all possible distance configurations on n points of diameter ≤ 1. This means:
$\varphi \in \Phi_n$ if $\varphi(v_1, ..., v_n)$ is a formula of the form

$$\max_{1 \leq i,j \leq n} |d(v_i, v_j) - r_{ij}|$$

where the matrix $(r_{ij})_{1\leq i,j\leq n}$ is a distance matrix of some finite metric space of diameter ≤ 1. Then $\varphi(a_1,\ldots,a_n) = 0$ if and only if $\{a_1,\ldots,a_n\}$ has distance matrix $(r_{ij})_{1\leq i,j\leq n}$.

Remark. A distance matrix $\left(r_{ij}\right)_{1\leq i,j\leq n}$ is a two-dimensional array containing distances (all bounded by 1) as elements such that $r_{ii} = 0$, $r_{ij} \geq 0$, $r_{ij} = r_{ji}$ and $r_{ik} + r_{kj} \geq r_{ij}$, for all $i, j, k \in \{1, 2, ..., n\}$.

Let us introduce the following notation: for $\varphi \in \Phi_{n+1}$, let $\varphi\mid_n$ be the restriction of φ to the first n variables, i.e., take the maximum up to n instead of $n+1$.

Theorem 5.1. *(See [16], p.1616). For every $\varphi \in \Phi_{n+1}$ and $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $a_1, ..., a_n \in \mathcal{U}$ satisfy $\varphi\mid_n (a_1, ..., a_n) < \delta$, then there exists $a_{n+1} \in \mathcal{U}$ such that $\varphi(a_1, ..., a_n, a_{n+1}) \leq \epsilon$.*

Such result goes back to Vershik's 2002 preprint "Distance matrices, random metrics and Urysohn space" [18], and is phrased thus in Usvyatsov's 2007 "Generalized Vershik's Theorem and generic Metric Structures" [17], whose introduction yields more details. This is an extension property (every finite metric subspace can be extended by one extra point in any compatible configuration), but in an approximate way only.

In short,

$$\sigma(\epsilon, \delta) : \forall v_1, ..., v_n \exists y(\varphi\mid_n (v_1, ..., v_n) < \delta \rightarrow \varphi(v_1, ..., v_n, y) \leq \epsilon)$$

Usvyatsov noted that it can be written in continuous logic thus:

$$\sup_{v_1,...,v_n} \inf_y \min\left(\frac{\epsilon}{1-\delta}\left(1 - \varphi\mid_n (v_1, ..., v_n)\right), \varphi(v_1, ..., v_n, y)\right) \leq \epsilon$$

Note that ϵ, and hence δ are both fixed in that instance of the axiom scheme, that is, for each $\epsilon > 0$ and accompanying δ we have a statement $\sigma(\epsilon, \delta)$.

Let $T_{\mathcal{U}}$ be the collection of all the conditions of that form.

According to [16], the only separable complete model of $T_{\mathcal{U}}$ is $\mathcal{U}$.

We will explain the following ground results:

5.2.1 $T_{\mathcal{U}}$ is $\aleph_0$-categorical.

This means that $T_{\mathcal{U}}$ has only one separable complete model (up to isometry), which is $\mathcal{U}$ according to Theorem 5.2.

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be models of $T_{\mathcal{U}}$, and each one having density character [1] $\aleph_0$. Let M_1 and N_1 be countable dense subsets of $\mathcal{M}$ and $\mathcal{N}$, respectively. We have $M_1 = \{x_0, x_1, ...\}$ and $N_1 = \{y_0, y_1, ...\}$ by listing. So, we will build a sequence of finite isometries $f_0 \subseteq f_1 \subseteq f_2 \subseteq ...$ such that for all x, y in the domain of f_k, we have

$$d^{\mathcal{N}}(f_k(x), f_k(y)) = d^{\mathcal{M}}(x, y) \qquad (*)$$

Let A and B finite subspaces of $\mathcal{M}$ and $\mathcal{N}$, respectively, and f an isometry between them.

Let $f_0 = \emptyset$, $f_1 = f$. Assume we have defined $f_2, \ldots, f_k$. We want to define f_{k+1}. Then $k+1$ can be either even or odd.

Case $k+1 = 2i+1$: We make sure that x_i is in the domain of f_{k+1}.

If x_i is in the domain of f_k, let $f_{k+1} = f_k$. If not, let $\alpha_1, \ldots, \alpha_m$ list the domain of f_k and let $\varphi(\alpha_1, \ldots, \alpha_m, x_i) = 0$ where $\varphi \in \Phi_{m+1}$. Since $\mathcal{N} \models T_{\mathcal{U}}$, for $\epsilon = 1/2$ there is $\delta(\epsilon)$ such that $\sigma(\epsilon, \delta)$ holds, and then we can find $z_1 \in \mathcal{N}$ such that $\varphi|_m(f_k(\alpha_1), \ldots, f_k(\alpha_m)) < \delta \to \varphi(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_1) < \epsilon$. Since $\varphi|_m(\alpha_1, \ldots, \alpha_m) = 0$, so $\varphi|_m(f_k(\alpha_1), \ldots, f_k(\alpha_m)) = 0$, and then

$$\varphi(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_1) < \frac{1}{2}.$$

Now let $\varphi^* \in \Phi_{m+2}$ such that $\varphi^*(\alpha_1, \ldots, \alpha_m, x_i, x_i) = 0$. Since $\mathcal{N} \models T_{\mathcal{U}}$ and for $\epsilon = 1/4$, repeating above argument we find $z_2 \in \mathcal{N}$ such that

$$\varphi^*(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_1, z_2) < 1/4.$$

In particular, $d(z_1, z_2) < 1/4$, but also $\varphi(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_2) < 1/4$. In general, find z_{t+1} such that $\varphi^*(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_t, z_{t+1}) < 1/2^{t+1}$. Hence the sequence (z_t) in $\mathcal{N}$ is a Cauchy sequence since $d(z_t, z_{t+1}) < 1/2^{t+1}$. Let its limit be z, so $\varphi(f_k(\alpha_1), \ldots, f_k(\alpha_m), z) = 0$ since $\varphi(f_k(\alpha_1), \ldots, f_k(\alpha_m), z_t) < 1/2^t$.

Extend f_k to f_{k+1} by taking $f_{k+1}(x_i) = z$. Then f_{k+1} satisfies $(*)$.

1 A character density of X is the minimum cardinality of the dense subsets of it.

Case $k+1 = 2i+2$*:* We make sure that y_i is in the range of f_{k+1}. Work as above by letting $\beta_1, \ldots, \beta_m$ list the range of f_k, letting $\varphi(\beta_1, \ldots, \beta_m, y_i) = 0$ and finding $w \in \mathcal{M}$ such that $\varphi(f_k^{-1}(\beta_1), \ldots, f_k^{-1}(\beta_m), w) = 0$, so take $f_{k+1}(w) = y_i$. Then f_{k+1} satisfies $(*)$.

Let $\hat{f} = \bigcup_{k=0}^{\infty} f_k$. We have constructed a isometry between countable dense subsets of $\mathcal{M}$ and $\mathcal{N}$ that extends f and satisfy $(*)$. In addition, we obtain an isometry F between $\mathcal{M}$ and $\mathcal{N}$ from $\hat{f}$ by using completeness. Thus $\mathcal{M} \cong \mathcal{N}$ and $T_{\mathcal{U}}$ is $\aleph_0$-categorical.

□

This proof is the so-called "back-and-forth" method.

5.2.2 $T_{\mathcal{U}}$ admits quantifier elimination.

Definition. ([2], Definition 13.1) An L-formula $\varphi(v_1, ..., v_n)$ is *approximable in a theory T by quantifier-free formulas* if for every $\epsilon > 0$ there is a quantifier-free L-formula $\psi_\epsilon(v_1, ..., v_n)$ such that for all $\mathcal{M} \models T$ and all $a_1, ..., a_n \in M$, one has

$$|\varphi^{\mathcal{M}}(a_1, ..., a_n) - \psi_\epsilon^{\mathcal{M}}(a_1, ..., a_n)| \leq \epsilon.$$

An L-theory T admits *quantifier elimination* if every L-formula is approximable in T by quantifier-free formulas.

Now, let us show that $T_{\mathcal{U}}$ has quantifier elimination.

Proof. (We quote verbatim [7]). Given $a = (a_1, ..., a_n) \in \mathcal{U}$ and $C \subseteq \mathcal{U}$ note that the quantifier-free type [2] of a over C is entirely determined by the following quantifier-free set of formulas (see below):

$$\{d(v_i, v_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \cup \{d(v_i, c) = d(a_i, c) : 1 \leq i \leq n, c \in C\}.$$

We use Proposition 13.2 and Lemma 13.5 in [2].

Proposition 5.2. *(Proposition* 13.2*). Let $\varphi(v_1, ..., v_n)$ be an L-formula. The following statements are equivalent.*

2 In a quantifier-free type its L-conditions involved have quantifier-free formulas.

(1) φ is approximable in T by quantifier-free formulas;

(2) Whenever we are given

- *models $\mathcal{M}$ and $\mathcal{N}$ of T;*
- *substructures $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$;*
- *an isomorphism Φ from $\mathcal{M}_0$ onto $\mathcal{N}_0$; and*
- *elements $a_1, \cdots, a_n$ of $\mathcal{M}_0$;*

we have

$$\varphi^{\mathcal{M}}(a_1, \cdots, a_n) = \varphi^{\mathcal{N}}(\Phi(a_1), \cdots, \Phi(a_n)).$$

Moreover, for the implication (2) $\Rightarrow$ (1) it suffices to assume (2) only for the cases in which $\mathcal{M}_0$ and $\mathcal{N}_0$ are finitely generated.

Lemma 5.3. *(Lemma 13.5). Suppose that T is an L-theory and that every restricted L-formula of the form $\inf_v \varphi$, with φ quantifier-free, is approximable in T by quantifier-free formulas. Then T admits quantifier elimination.*

Fix a quantifier-free formula $\varphi(x, v_1, ..., v_n)$. We want to show that the formula $\inf_x \varphi(y, \bar{v})$ is approximable in $T_{\mathcal{U}}$ by quantifier-free formulas. Fix $\mathcal{M}, \mathcal{N} \models T_{\mathcal{U}}$, substructures $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$, an isomorphism Φ from $\mathcal{M}_0$ onto $\mathcal{N}_0$, and elements $a_1, ..., a_n \in \mathcal{M}_0$. It suffices to show that for any $\epsilon > 0$,

$$\inf_x^{\mathcal{N}} \varphi(x, \Phi(a_1), ..., \Phi(a_n)) < \inf_x^{\mathcal{M}} \varphi(x, a_1, ..., a_n) + \epsilon.$$

Let $b \in \mathcal{M}$ be such that $\varphi^{\mathcal{M}}(b, a) < \inf_x^{\mathcal{M}} \varphi(x, \bar{a}) + \epsilon$. Note that, since Φ is an isometry, the space $X = \{x, \Phi(a_1), ..., \Phi(a_n)\}$ with $d(\Phi(a_i), \Phi(a_j)) = d(a_i, a_j)$ and $d(x, \Phi(a_i)) = d(b, a_i)$ is a metric space.
Therefore the type

$$\{d(x, \Phi(a_i)) = d(b, a_i) : 1 \leq i \leq n\}$$

is realized by some $c \in \mathcal{N}$ ($\mathcal{N} \models T_{\mathcal{U}}$ and the reasoning in 5.2.1 shows that the distance matrix is realized in $\mathcal{N}$). Then Φ extends to an isomorphism from $\{a_1, ..., a_n, b\}$ to $\{\Phi(a_1), ..., \Phi(a_n), c\}$. Therefore $\varphi(c, \Phi(\bar{a})) = \varphi(b, \bar{a})$, since $\varphi(x, \bar{v})$ is quantifier-free. It follows that

$$\inf_x^{\mathcal{N}} \varphi(x, \Phi(\bar{a})) \leq \varphi(c, \Phi(\bar{a})) = \varphi(b, \bar{a}) < \inf_x^{\mathcal{M}} \varphi(x, \bar{a}) + \epsilon,$$

as desired.

□

5.2.3 $T_{\mathcal{U}}$ is the model completion of the empty L-theory.

A model completion of a theory T_1 is a theory T_2 satisfying three properties:

(i) Every model of T_1 can be extended to a model of T_2, and every model of T_2 can be extended to a model of T_1.

(ii) T_2 is a model complete, that is, every embedding between models of T_2 is elementary.

(iii) If $\mathcal{M}$ is a model of T_1, then T_2 plus the atomic conditions which hold in $\mathcal{M}$ form a complete theory.

By unravelling that, the statement about the theory of the Urysohn space means: every complete metric space can be embedded in a model of $T_{\mathcal{U}}$ (which need not be $\mathcal{U}$ itself) and such embedding is unique; plus every embedding between models of $T_{\mathcal{U}}$ is elementary [3], i.e., $T_{\mathcal{U}}$ has universal axioms, that is, axioms of the form sup $-$ sup $- \ldots -$ sup-formula.

5.2.4 $T_{\mathcal{U}}$ is the theory of existentially closed metric spaces of diameter bounded by 1.

This is a concept borrowed from algebra: A field K is *algebraically closed* if every polynomial with coefficients in K and root in a extension of K already has a root in K itself. In classical logic, a structure $\mathcal{M}$ is *existentially closed* if any formula with parameters in $\mathcal{M}$ which is satisfied in some extension of $\mathcal{M}$ already is satisfied in $\mathcal{M}$ itself.

For continuous logic, this is written exactly the same way in Definition 18.15 in [2], but with a difference: the inf quantifier is not an existence quantifier. Then $\mathcal{U}$ is existentially closed because of quantifier elimination, and parameters in the approximate formula form a finite metric subspace.

3 An *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ is a function F formula value preserving.

5.2.5 Types are completely determined by distances.

First, let us define what a type is.
If $\mathcal{M}$ is a L-structure, $A \subset \mathcal{M}$, and $\bar{a} \in \mathcal{M}$, we say that the set $\{L(A) - \textit{conditions}\ E(\bar{v}) : \mathcal{M} \models E(\bar{a})\}$ is a *n-type* and we denote it by $tp^{\mathcal{M}}(\bar{a}/A)$.

Thus, we have

$$tp^{\mathcal{U}}(\bar{a}/C) = \{d(v_i, v_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \cup \{d(v_i, c) = d(a_i, c) : 1 \leq i \leq n, c \in C\}.$$

Now, let us see an interesting fact.

Fact. $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$ if and only if $(\mathcal{M}_A, \bar{a}) \equiv (\mathcal{M}_A, \bar{b})$ [4].

Since $tp^{\mathcal{M}}(\bar{a}) = \{$ condition $E(\bar{v})$ in $L(A) : \mathcal{M} \models E(\bar{a})\}$.
Assume $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$. Let φ be a formula and let $k = \varphi^{\mathcal{M}}(\bar{a}) \in [0, 1]$. Take condition $E(\bar{v}) : |\varphi(\bar{v}) - k| = 0$, so $E(\bar{v}) \in tp^{\mathcal{M}}(\bar{a})$, so $E(\bar{v}) \in tp^{\mathcal{M}}(\bar{b})$, so $\mathcal{N} \models E(\bar{b})$, so $|\varphi^{\mathcal{N}}(\bar{b}) - k| = 0$, then $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\bar{b})$. Do that for every φ in L to obtain $(\mathcal{M}, \bar{a}) \equiv (\mathcal{N}, \bar{b})$.
Conversely, assume $(\mathcal{M}, \bar{a}) \equiv (\mathcal{N}, \bar{b})$.
If $E(\bar{v}) \in tp^{\mathcal{M}}(\bar{a})$ then $E(\bar{v})$ is of the form $\varphi(\bar{v}) = k$, so $\varphi^{\mathcal{M}}(\bar{a}) = k$, so $\varphi^{\mathcal{N}}(\bar{b}) = k$, then $E(\bar{v}) \in tp^{\mathcal{M}}(\bar{b})$ (respectively for $\varphi(\bar{v}) \leq k$ and $\varphi(\bar{v}) \geq k$).

Then types are completely determined by distances in $\mathcal{U}$, because homogeneity of $\mathcal{U}$ ensures that if $\bar{a}, \bar{b}$ have the same distance metric with regard to a finite set $\{a_1, \ldots, a_n\}$, then there is an isometry taking $\bar{a}$ to $\bar{b}$, so $(\mathcal{U}, \bar{a}) \equiv (\mathcal{U}, \bar{b})$.

Finally, we can state the definition of an ω-saturated structure: $\mathcal{M}$ is ω-saturated if for every finite $A \subseteq \mathcal{M}$, any type p which is finitely realized in $\mathcal{M}$ is also realized in $\mathcal{M}$. By p finitely realized in $\mathcal{M}$ we mean: every finite subset of p is realized in $\mathcal{M}$. It follows that the inf quantifier works as $\exists$ in ω-saturated $\mathcal{M}$, because if $\mathcal{M} \models \inf_x \varphi(x) = 0$ then the type $\{\varphi(x) \leq \frac{1}{n} : n \in \mathbb{N}\}$ is finitely realized in $\mathcal{M}$, and so it is realized in $\mathcal{M}$.

4 $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent ($\mathcal{M} \equiv \mathcal{N}$), if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ.

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